A dynamic Duverger's Law
Supplementary appendix

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Abstract

This appendix contains the proofs for Propositions 1 and 2 from 'A dynamic Duverger's Law'. It also contains further description of the data used in that paper, as well as regression results in which total party movements are disaggregated into party entries and exits.

A Proofs of Propositions 1 and 2

Proof of Proposition 1. Note that (1) implies that under proportional representation, forming (or maintaining, since $\overline{c} > \underline{c}$) a party is uniquely stage optimal in preference state s_0 for party j, irrespective of whether interest group $-j$ is represented by a party. Also, since $\overline{p} > \frac{1}{3}$, (1) implies that $\overline{c} \leq \overline{p}[\overline{u} - u]$, so that forming (or maintaining, since $\overline{c} > \underline{c}$) a party is uniquely stage optimal in preference state s_j for party j, irrespective of whether interest group $-j$ is represented by a party. Finally, since $\overline{c} > \underline{c}$, it follows that, for any state (s, ϕ) and any equilibrium σ^*, $V_j(s, \phi \cup \{j\}; \sigma^*) \geq V_j(s, \phi; \sigma^*)$. Hence, in any equilibrium under proportional representation, it must be that $\sigma_j^*(s, \phi) = 1$ for all states such that $s \in \{s_0, s_j\}$.

It remains only to determine interest groups' equilibrium actions in preference state s_{-j}. Fix an equilibrium σ^* and consider a state (s_{-j}, ϕ) such that $j \in \phi$. If interest group j disbands its party, its payoff is

$$V_j(s_{-j}, \phi; \sigma^*) = (1 - \overline{p})u + \overline{p}\underline{u} + \delta \mathbb{E} V_j(s', \{-j\}; \sigma^*)$$

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If instead interest group j maintains its party, let $V^d(s_{-j}, \phi; \sigma^*)$ be its payoff. We have that

$$V_j^d(s_{-j}, \phi; \sigma^*) = \underline{p}\overline{u} + pu + \overline{p}\underline{u} - \underline{c} + \delta \mathbb{E} V_j(s', \{-j, j\}; \sigma^*).$$

By our results from above, we have that, for any $s \in \{s_0, s_j\}$,

$$V_j(s, \{-j\}; \sigma^*) = V_j(s, \{-j, j\}; \sigma^*) - [\overline{c} - \underline{c}],$$

so that $V_j(s_{-j}, \phi; \sigma^*) > V_j^d(s_{-j}, \phi; \sigma^*)$ if and only if (2) holds. Note that (2) also implies that in state (s_{-j}, ϕ) such that $j \notin \phi$, interest group j strictly prefers not to form a party. Hence, for any equilibrium σ^* under proportional representation, we have that $\sigma^* = \sigma^{PR}$. □

Proof of Proposition 2. Fix any equilibrium σ^*. First, note that since $\beta \geq 0$, under plurality as under proportional representation, (1) implies that maintaining an existing party is uniquely stage optimal in preference state s_0 for interest group j, irrespective of whether interest group $-j$ is represented by a party. Hence, by the arguments in the proof of Proposition 1, $\sigma_j^*(s_0, \phi) = 1$ whenever $j \in \phi$. Second, since $\alpha \geq 0$, (1) also implies that $\sigma_j^*(s_j, \phi) = 1$ whenever $j \in \phi$. Third, since no new party faces entry penalty β following entry when $\phi = \emptyset$, (1) also ensures that $\sigma_j^*(s, \emptyset) = 1$ is uniquely optimal when $s \in \{s_0, s_j\}$.

Now consider state $(s_0, \{-j\})$ and equilibrium σ^*. If interest group j does not form a party, its payoff is

$$\frac{1 + \overline{p}}{2} u + \frac{1 - \overline{p}}{2} \underline{u} + \delta \mathbb{E} V_j(s', \{-j\}; \sigma^*),$$

while if interest group j forms a party, its payoff is

$$\left(\frac{1 - \overline{p}}{2} - \beta \right) \overline{u} + \overline{p} u + \left(\frac{1 - \overline{p}}{2} + \beta \right) \underline{u} - \overline{c} + \delta \mathbb{E} V_j(s', \{-j, j\}; \sigma^*).$$

Hence, interest group j does not form a party if and only if

$$\begin{aligned} \overline{c} - \left[\frac{1 - \overline{p}}{2} [\overline{u} - u] - \beta [\overline{u} - \underline{u}] \right] &\geq \delta \mathbb{E} \Big[V_j(s', \{-j, j\}; \sigma^*) - V_j(s', \{-j\}; \sigma^*) \Big] \\ &\equiv \delta \mathbb{E} \Delta V_j(s'; \sigma^*) \end{aligned} \tag{A1}$$

Consider state (s_{-j}, ϕ) such that $j \in \phi$ and such that $\sigma_{-j}^*(s_{-j}, \phi) = 1$. If interest group j maintains its party, its payoff is

$$(\underline{p} - \alpha + \beta \mathbb{I}_{-j \notin \phi}) \overline{u} + pu + (\overline{p} + \alpha - \beta \mathbb{I}_{-j \notin \phi}) \underline{u} - \underline{c} + \delta \mathbb{E} V_j(s', \{-j, j\}; \sigma^*),$$

while if interest group j disbands its party, its payoff is

$$(1-\overline{p})u + \overline{p}\underline{u} + \delta \mathbb{E} V_j(s', \{-j\}; \sigma^*).$$

Hence, under profile $\underline{\sigma}^{PL}$, it must be that

$$\underline{c} - \underline{p}[\overline{u} - u] + (\alpha - \beta)[\overline{u} - \underline{u}] \geq \delta \mathbb{E} \Delta V_j(s'; \underline{\sigma}^{PL}), \tag{A2}$$

while under profile $\overline{\sigma}^{PL}$, it must be that

$$\underline{c} - \underline{p}[\overline{u} - u] + \alpha[\overline{u} - \underline{u}] \leq \delta \mathbb{E} \Delta V_j(s'; \overline{\sigma}^{PL}). \tag{A3}$$

Fix a state (s_j, ϕ) such that $j \notin \phi$. Under $\underline{\sigma}^{PL}$, (1) ensures that the stage payoffs of interest group j are strictly positive when it forms a party, so that, by an argument in the proof of Proposition 1, $\underline{\sigma}^{PL}(s_j, \phi) = 1$ is optimal. Under $\overline{\sigma}^{PL}$, interest group j forms a party in state (s_j, ϕ) with $j \notin \phi$ if and only if

$$\overline{p}[\overline{u} - u] - \overline{c} + (\alpha - \beta)[\overline{u} - \underline{u}] \geq -\delta \mathbb{E} \Delta V_j(s'; \overline{\sigma}^{PL}). \tag{A4}$$

Note that (A2), along $\underline{\sigma}_{-j}^{PL}(s_{-j}, \emptyset) = 1$ and the fact that $\overline{c} > \underline{c}$, implies that $\underline{\sigma}_j^{PL}(s_{-j}, \emptyset) = 0$ is optimal. Since the profile $\overline{\sigma}^{PL}$ is specified in all states except $(s_{-j}, \emptyset)$, a simple computation verifies whether either $\overline{\sigma}_j^{PL}(s_{-j}, \emptyset) = 0$ or $\overline{\sigma}_j^{PL}(s_{-j}, \emptyset) = 0$ are optimal. Actions in this state are irrelevant when verifying equilibrium incentives, since under $\overline{\sigma}^{PL}$ it can be reached only following deviations by two interest groups.

Hence, the relevant incentive constraints under $\underline{\sigma}^{PL}$ are (A1) and (A2), while the relevant incentive constraints under $\overline{\sigma}^{PL}$ are (A1), (A3) and (A4). These can be further simplified through computation. First, note that

$$\begin{aligned}
\Delta V_j(s_j; \overline{\sigma}^{PL}) &= \overline{c} - \underline{c} + \beta[\overline{u} - \underline{u}], \\
\Delta V_j(s_j; \underline{\sigma}^{PL}) &= \overline{c} - \underline{c}, \\
\Delta V_j(s_{-j}; \underline{\sigma}^{PL}) &= 0,
\end{aligned}$$

so that we have that

$$\Delta V_j(s_{-j}; \overline{\sigma}^{PL}) = \frac{1}{1 - \delta \frac{1-q}{2}} \left[\underline{p}[\overline{u} - \underline{u}] - \alpha[\overline{u} - \underline{u}] - \underline{c} + \delta q \Delta V_j(s_0; \overline{\sigma}^{PL}) + \delta \frac{1-q}{2} \Delta V_j(s_j; \overline{\sigma}^{PL}) \right],$$

and that

$$\Delta V_j(s_0;\overline{\sigma}^{PL}) = \frac{1}{1-\delta q}\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c} + \delta\frac{1-q}{2}\Delta V_j(s_{-j};\overline{\sigma}^{PL}) + \delta\frac{1-q}{2}\Delta V_j(s_j;\overline{\sigma}^{PL})\right].$$

Further computation yields that

$$\begin{aligned}\delta\mathbb{E}\Delta V_j(s';\overline{\sigma}^{PL}) = \frac{1}{1-\delta\frac{1+q}{2}}&\left[\delta\frac{1-q}{2}\left[\underline{p}[\overline{u}-u] - \alpha[\overline{u}-\underline{u}] - \underline{c}\right]\right.\\ &\left.+\delta q\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] + \delta\frac{1-q}{2}\left[\overline{c}-\underline{c}+\beta[\overline{u}-\underline{u}]\right]\right].\end{aligned}$$

Similarly,

$$\Delta V_j(s_0;\underline{\sigma}^{PL}) = \frac{1}{1-\delta q}\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c} + \delta\frac{1-q}{2}\Delta V_j(s_j;\underline{\sigma}^{PL})\right],$$

and further computation yields that

$$\delta\mathbb{E}\Delta V_j(s';\underline{\sigma}^{PL}) = \frac{1}{1-\delta q}\left[\delta q\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] + \delta\frac{1-q}{2}\left[\overline{c}-\underline{c}\right]\right].$$

Evaluated at $\overline{\sigma}^{PL}$, (A1) can be rewritten as

$$\beta[\overline{u}-\underline{u}] \geq \frac{1-\delta\frac{1-q}{2}}{1-\delta q}\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] - [\overline{c}-\underline{c}] + \frac{\delta\frac{1-q}{2}}{1-\delta q}\left[\underline{p}[\overline{u}-\underline{u}] - \alpha[\overline{u}-\underline{u}] - \underline{c}\right], \tag{A5}$$

while evaluated at $\underline{\sigma}^{PL}$, it can be rewritten as

$$\beta[\overline{u}-\underline{u}] \geq \frac{1}{1-\delta q}\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] - \frac{1-\delta\frac{1+q}{2}}{1-\delta q}\left[\overline{c}-\underline{c}\right]. \tag{A6}$$

A straightforward computation verifies that, for any α, the righthand side of (A6) is strictly larger than the righthand side of (A5), so that (A5) holds whenever (A6) holds.

Also, (A2) can be rewritten as

$$\alpha[\overline{u}-\underline{u}] \geq \underline{p}[\overline{u}-u] - \underline{c} + \beta[\overline{u}-\underline{u}] + \frac{1}{1-\delta q}\left[\delta q\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] + \delta\frac{1-q}{2}\left[\overline{c}-\underline{c}\right]\right], \tag{A7}$$

while (A3) can be rewritten as

$$\alpha[\overline{u}-\underline{u}] \leq \underline{p}[\overline{u}-u] - \underline{c} + \frac{1}{1-\delta q}\left[\delta q\left[\frac{1-\overline{p}}{2}[\overline{u}-u] - \underline{c}\right] + \delta\frac{1-q}{2}\left[\overline{c}-\underline{c}+\beta[\overline{u}-\underline{u}]\right]\right]. \tag{A8}$$

Finally, since the righthand side of (A4) is increasing in α, it can be shown by computation to hold for all α if and only if

$$\beta[\overline{u}-\underline{u}] \leq \overline{p}[\overline{u}-u]-\overline{c}+\frac{\delta}{1-\delta}\left[\frac{1-\overline{p}}{2}[\overline{u}-u]-\underline{c}\right]+\frac{\delta(1-q)}{1-\delta}\frac{p-\underline{p}}{2}[\overline{u}-u], \tag{A9}$$

That (A6) holds follows since $\beta \geq \underline{\beta}$, and that (A9) holds follows since $\beta \leq \overline{\beta}$. Hence, conditions (A6) and (A7) are sufficient for $\underline{\sigma}^{PL}$ to be an equilibrium, while (A6), (A8) and (A9) are sufficient for $\overline{\sigma}^{PL}$ to be an equilibrium. Let $\check{\alpha}$ be the unique value of α such that (A7) holds as an equality and define $\underline{\alpha} = \max\{\min\{\underline{p}-\beta, \check{\alpha}\}, 0\}$. Similarly, let $\hat{\alpha}$ be the unique value of α such that (A8) holds as an equality and define $\overline{\alpha} = \min\{\max\{0, \hat{\alpha}\}, \underline{p}-\beta\}$. Hence, given any β satisfying (A6), $\underline{\sigma}^{PL}$ is an equilibrium if $\alpha > \underline{\alpha}$, while $\overline{\sigma}^{PL}$ is an equilibrium if $\alpha < \overline{\alpha}$. These are sufficient conditions only, since our definition of $\underline{\alpha}$ and $\overline{\alpha}$ embeds the cases when these equilibria fails to exits. Furthermore, (A7) and (A8) imply that $\underline{\alpha} \geq \overline{\alpha}$, where the inequality is strict whenever $\underline{\alpha}, \overline{\alpha} \in (0, \underline{p}-\beta)$. □

B Additional data descriptions and empirical results

Table B.1: Data

Country	Years	Elections	Country	Years	Elections
Australia	1993-2001	4	Luxembourg	1945-2004	14
Austria	1945-2008	20	Malaysia	1959-1999	11
Belgium	1978-2007	18	Malta	1945-2003	16
Bermuda	1989-1998	3	Mauritius	1995-2000	2
Bolivia	1989-2002	6	Mexico	1997-2000	5
Botswana	1999	1	Netherlands	1948-2006	18
Bulgaria	1994-2005	4	New Zealand	1946-1999	22
Canada	1945-2000	26	Norway	1977-2005	8
Costa Rica	1953-2002	13	Poland	1991-2007	12
Cyprus	1991-1996	2	Portugal	1975-2005	12
Czech Republic	1996-2008	7	Romania	1992-2004	8
Estonia	1992-2007	5	Russia	1995-1999	2
Finland	1999-2007	3	Slovakia	1994-1998	2
France	1988-2007	5	South Africa	1994-1999	4
Germany	1990-2009	6	Spain	1977-2008	20
Greece	1946-2007	20	Sweden	1944-2006	20
Hungary	1990-2006	5	Switzerland	1947-2007	16
Iceland	1959-2007	13	Trin. & Tobago	1966-2002	10
Ireland	1948-1997	16	Turkey	1999-2002	6
Israel	1949-2003	16	United Kingdom	1945-2005	16
Italy	1948-2006	14	United States	1986-2000	16
Latvia	1993-2006	5	Venezuela	1958-1988	7

Notes: All data comes from the Constituency-Level Elections Dataset.

Table B.2: Dynamic tests of Duverger's Law: Entry

Variable	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Majoritarian dummy	-0.21**	-0.34***	-0.28***						
	(0.08)	(0.09)	(0.09)						
Average district magnitude				0.06**	0.11**	0.08***			
				(0.03)	(0.03)	(0.03)			
Average disproportionality index							-0.72*	-1.01***	-0.11
							(0.43)	(0.32)	(0.75)
Decade, regional, and district number controls included?	N	Y	Y	N	Y	Y	N	Y	Y
Flexibly controlled for number of parties?	N	N	Y	N	N	Y	N	N	Y
R^2	0.03	0.18	0.30	0.04	0.20	0.31	0.02	0.17	0.28
Number of observations	411	411	411	411	411	411	411	411	411

Notes: Dependent variable is total weighted number of party entries as computed with a 5% inclusion threshold. Majoritarian dummy is obtained from **?**. Average disproptionality Index for a given country is constructed by averaging the disproportionality Index for each election in the sample for each country. Flexible control for the number of districts and parties is achieved by including sixth order polynomials in those variables and in the log of those variables (24 covariates). All continuous variables are specified in logarithms. To conserve data, dependent variable is transformed as $\log(1+x)$. Robust standard errors clustered by country are presented in parentheses. *** - 1% significance level, ** - 5% significance level, * - 10% significance level

Table B.3: Dynamic tests of Duverger's Law: Exit

Variable	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Majoritarian dummy	-0.19**	-0.29***	-0.30***						
	(0.08)	(0.10)	(0.10)						
Average district				0.06*	0.07**	0.07**			
magnitude				(0.03)	(0.03)	(0.03)			
Average disproportionality							-0.50	-0.71*	-0.16
index							(0.547	(0.43)	(0.71)
Decade, regional, and district number controls included?	N	Y	Y	N	Y	Y	N	Y	Y
Flexibly controlled for number of parties?	N	N	Y	N	N	Y	N	N	Y
R^2	0.03	0.15	0.22	0.03	0.15	0.22	0.01	0.14	0.20
Number of observations	411	411	411	411	411	411	411	411	411

Notes: Dependent variable is total weighted number of party exits as computed with a 5% inclusion threshold. Majoritarian dummy is obtained from **?**. Average disproptionality Index for a given country is constructed by averaging the disproportionality Index for each election in the sample for each country. Flexible control for the number of districts and parties is achieved by including sixth order polynomials in those variables and in the log of those variables (24 covariates). All continuous variables are specified in logarithms. To conserve data, dependent variable is transformed as $\log(1+x)$. Robust standard errors clustered by country are presented in parentheses. *** - 1% significance level, ** - 5% significance level, * - 10% significance level