Fixed Point Approaches to the Proof of the Bondareva-Shapley Theorem*

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Abstract

We provide two new proofs of the Bondareva-Shapley theorem, which states that the core of a transferable utility cooperative game has a nonempty core if and only if the game is balanced. Both proofs exploit the fixed points of self-maps of the set of imputations, applying elementary existence arguments typically associated with non-cooperative games to cooperative games.

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1 Introduction

The celebrated Bondareva-Shapley Theorem (Bondareva (1962, 1963); Shapley (1967)) shows that balancedness is both necessary and sufficient for the existence of the core of a transferable utility (TU) cooperative game. In this paper, we provide two new proofs of this theorem which rely on elementary fixed point methods. Our aim is to approach the problem of core existence through analogies to common proofs of the existence of Nash equilibria in noncooperative games, where coalitional blocking of imputations in cooperative games stands for best-responses to strategy profiles in noncooperative games. While a Nash equilibrium is a strategy profile that *is* a best-response to itself, a core imputation is *not* blocked by any other imputation. Therefore, both our proofs start by assuming that some balanced TU

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game fails to have a core and derives a contradiction: only in this case does the blocking relation yield a well-defined "best-response" to each imputation.[1]

In our first proof, the analogy is to establishing the existence of Nash equilbria through fixed points of best-response correspondences (Nash (1950)). If a *TU* game has no core and every imputation is blocked by some coalition, then we can construct payoffs for blocking coalitions and define a (non-empty valued) correspondence from the set of imputations into itself. The existence of a fixed point of this correspondence, which corresponds to an imputation that is a convex combination of imputations that block it, is established through an elementary fixed point theorem for lower hemicontinuous correspondences due to Gale and Mas-Colell (1975).[2] Furthermore, this fixed point identifies a balanced collection of blocking coalitions, which contradicts the balancedness of the game.

In our second proof, the analogy is to establishing the existence of Nash equilibria through an application of Brouwer's fixed point theorem to a continuous function on the set of mixed strategies of the game that has tâtonnement-type properties: it increases the use of pure strategies that are best-responses and decreases the use of those that are not (Nash (1951)). By adapting a construction from Zhou (1994), we provide a continuous function on the set of imputations of the game that tends to increase the payoffs of blocking coalitions and decrease the payoffs of non-blocking coalitions, and, as in our first proof, its fixed point also identifies a balanced collection of blocking coalitions. Zhou (1994) constructs a similar function as an intermediate step in the proof of an intersection of open covers theorem which is closely related to the K-K-M-S theorem (see Scarf (1967), Shapley (1967), Ichiishi (1981), Kannai (1992), Shapley and Vohra (1991), Krasa and Yannelis (1994), Komiya (1994) and Herings (1997)), and which he then applies to provide an alternative proof of Scarf's (1967) theorem that all balanced nontransferable utility games have nonempty cores.[3]

The original proof of the Bondareva-Shapley theorem relies on duality results from linear programming. A second proof by Aumann (1989) establishes a connection between core existence and the minimax theorem for zero-sum games: given a balanced TU game, he constructs a zero-sum game whose mixed strategy equilibrium identifies a core imputation.

[1] A similar strategy can establish the existence of maximal elements of preference correspondences. See, e.g., Sonnenschein (1971) and Yannelis and Prabhakar (1983).

[2] The theorem of Gale and Mas-Colell (1975) requires correspondences with open graphs, but their proof will work for lower hemicontinuous correspondences, as pointed out by Yannelis and Prabhakar (1983, Theorem 3.4). This result has recently been significantly generalised by He and Yannelis (2017), whose result also provides a generalisation for Fan-Glicksberg's fixed point theorem for upper hemicontinuous correspondences and Browder's fixed point theorem for open fibers.

[3] It is well-known that every balanced *TU* game can be expressed as a balanced nontransferable utility game. Therefore, a nonemptiness result for the latter implies the nonemptiness of the core of a (*TU*) game.

Finally, another approach is based on the separating hyperplane theorem: see for example Osborne and Rubinstein (1994) and Peleg and Sudhölter (2003). Although these other proofs of the Bondareva-Shapley theorem are now well-known, we believe that our contribution is of independent interest and enhances our understanding of the core as a solution concept for cooperative games. First, as opposed to the proof of Aumann (1989), a notable feature of our proofs is that we do not borrow results from noncooperative games to establish the Bondareva-Shapley theorem, but instead we prove the theorem through the methods used to establish the existence of equilibria in noncooperative games. In doing so, we bring the tools, approach and level of difficulty of the proof of the Bondareva-Shapley theorem in line with what is familiar to all graduate students in economics. Second, our means of tackling the core existence problem for TU games, by studying the fixed points of mappings generated by the blocking relation, is general enough that it may prove useful for other problems in cooperative game theory.

2 Two Proofs of the Bondareva-Shapley Theorem

Given a set of players N, a *transferable utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$, where $\mathcal{N}$ is the set of all nonempty subsets of N. Let V be the set of imputations for this game, that is, the set of individually rational utilities attainable for the grand coalition, or

$$V = \{v \in \mathbb{R}^N : v_i \geq W(\{i\}) \text{ for all } i \in N \text{ and } \sum_{i \in N} v_i = W(N)\}.$$

Let V be endowed with the relative Euclidean topology. The *core* of the game W is defined as $\text{Core}(W) = \{v \in V | \sum_{i \in S} v_i \geq W(S) \text{ for all } S \in \mathcal{N}\}$. A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is *balanced* if there exists weights $\{\delta_S\}_{S \in \mathcal{B}}$ such that $\delta_S \geq 0$ for all $S \in \mathcal{B}$ and $\sum_{S \in \mathcal{B},\ i \in S} \delta_S = 1$ for all $i \in N$. The game W is *balanced* if for all balanced collections of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \delta_S W(S) \leq W(N)$.

Theorem (Bondareva-Shapley). *A TU game has a nonempty core if and only if it is balanced.*

We make a few remarks before giving our proofs of the theorem. First, given $0 < \epsilon < 1$, we normalize[4] the game such that $W(\{i\}) = \epsilon$ for all $i \in N$. To see this, fix any balanced game W, and consider another game $\tilde{W}$ such that $\tilde{W}(S) = W(S) + \sum_{i \in S} [\epsilon - W(\{i\})]$ for

[4] See McKinsey (1950) for the normalization of TU games.

all $S \in \mathcal{N}$ along with the isomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x)_i = x_i + \epsilon - W(\{i\})$ for all $i \in N$. Clearly, a coalition $S \in \mathcal{N}$ blocks imputation $v \in V$ if and only if coalition S also blocks imputation $f(v) \in f(V)$. Second, we expand the set of imputations associated to W such that

$$\hat{V} = \{v \in \mathbb{R}^N : v_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in N} v_i = W(N)\},$$

and modify the definition of the core such that

$$\widehat{\text{Core}}(W) = \{v \in \hat{V} | \sum_{i \in S} v_i \geq W(S) \text{ for all } S \in \mathcal{N}\}.$$

Clearly, because for all $v \in \hat{V} \setminus V$, there exists $i \in N$ such that $v_i < W(\{i\}) = \epsilon$, we have that $\widehat{\text{Core}}(W) = \text{Core}(W)$. Third, if the game is balanced, then $W(N) > 0$ (because the collection $\{\{i\} : i \in N\}$ is balanced, with weights such that $\delta_{\{i\}} = 1$). Finally, the necessity of balancedness for a nonempty core follows from standard arguments, so that we only prove its sufficiency.

A First Proof of the Bondareva-Shapley Theorem. Towards a contradiction, assume that W has an empty core. It follows that all $v \in \hat{V}$ are blocked by some coalition, and let $\mathcal{S}(v)$ denote the set of coalitions that block v. Define $\psi_S(v) \in \hat{V}$ such that, given any $v \in \hat{V}$ and any $S \in \mathcal{S}(v)$,

$$\psi_{S,i}(v) = \begin{cases} \frac{W(N)\max\{v_i,\epsilon\}}{\sum_{j \in S}\max\{v_j,\epsilon\}} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

where $\psi_S(v) = (\psi_{S,1}(v), \ldots, \psi_{S,n}(v))$. Define a correspondence $P : \hat{V} \twoheadrightarrow \hat{V}$ such that $P(v) = co\{\psi_S(v)\}_{S \in \mathcal{S}(v)}$, and note that P has nonempty and convex values.

Lemma 1. *There exists $\overline{v}$ such that $\overline{v} \in P(\overline{v})$.*

Given any imputation $v \in \hat{V}$ and coalition $S \in \mathcal{S}(v)$, the set of imputations that block v for S is open, so that P is not an upper hemicontinuous correspondence. However, P is lower hemicontinuous. Lemma 1 then follows from an application of the fixed point theorem of Gale and Mas-Colell (1975, Theorem), which, as re-stated in Yannelis and Prabhakar (1983, Theorem 3.4), guarantees that any non-empty and convex-valued lower semicontinuous correspondence on a nonempty, convex and compact subset of $\mathbb{R}^n$ has a fixed point.

The following lemma, in obvious contradiction with Lemma 1, completes the proof of the Bondareva-Shapley theorem.

Lemma 2. *For all $v \in \hat{V}$, $v \notin P(v)$.*

Proof. Suppose, toward a contradiction, that there exists $\overline{v} \in P(\overline{v})$, and fix $i \in N$. It follows that, for all $S \in \mathcal{S}(v)$, there exist $\lambda_S \geq 0$ with $\sum_{S \in \mathcal{S}(v)} \lambda_S = 1$ such that

$$\overline{v}_i = \sum_{S \in \mathcal{S}(v)} \lambda_S \psi_{S,i}(\overline{v}).$$

First, suppose that $\overline{v}_i \geq \epsilon$, so that

$$\overline{v}_i = \sum_{S \in \mathcal{S}(v), i \in S} \lambda_S \frac{W(N)\overline{v}_i}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}},$$

which because $\overline{v}_i > 0$ is equivalent to

$$\sum_{S \in \mathcal{S}(v), i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}} = 1.$$

Second, suppose that $\overline{v}_i < \epsilon$, so that

$$\begin{aligned}\overline{v}_i &= \sum_{S \in \mathcal{S}(v), i \in S} \lambda_S \frac{W(N)\epsilon}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}} \\ &< \epsilon,\end{aligned}$$

which, because $\epsilon > 0$ yields that

$$\sum_{S \in \mathcal{S}(v), i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}} < 1. \tag{1}$$

We next construct weights $\delta_S \geq 0$ for all $S \in \mathcal{N}$ so that the collection $\mathcal{N}$ is balanced. Let $I = \{\{i\} | i \in N\}$ be the set of all singleton coalitions. For all $S \in \mathcal{S}(\overline{v}) \setminus I$, let $\delta_S = {}^{\lambda_S W(N)}/_{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}}$. For all $i \in N$ such that $\overline{v}_i < \epsilon$ ((i.e. $\{i\} \in \mathcal{S}(\overline{v})$), set

$$\delta_{\{i\}} = 1 - \sum_{S \in \mathcal{S}(v) \setminus I, i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}},$$

Finally, for all other $S \in \mathcal{N}$, set $\delta_S = 0$. By construction, $\sum_{S \in \mathcal{N}, i \in S} \delta_S = 1$ for all $i \in N$, so that the collection $\mathcal{N}$ is balanced, and it follows that

$$\begin{aligned}\sum_{S\in\mathcal{N}}\delta_S W(S) &= \sum_{S\in\mathcal{S}(\overline{v})\setminus I}\delta_S W(S) + \sum_{\{i\}\in\mathcal{S}(\overline{v})}\delta_{\{i\}}W(\{i\}) \\ &= \sum_{S\in\mathcal{S}(\overline{v})\setminus I}\delta_S W(S) + \sum_{\{i\}\in\mathcal{S}(\overline{v})}\delta_{\{i\}}\epsilon \\ &> \sum_{S\in\mathcal{S}(\overline{v})}\delta_S\sum_{i\in S}\overline{v}_i \\ &= \sum_{i\in N}\overline{v}_i\sum_{S\in\mathcal{N},i\in S}\delta_S \\ &= W(N),\end{aligned}$$

contradicting the fact that the game is balanced. The inequality follows from the facts that (a) each coalition $S\in\mathcal{S}(\overline{v})\setminus I$ blocks $\overline{v}$, (b) each $\{i\}\in\mathcal{S}(\overline{v})$ is such that $\overline{v}_i<\epsilon$ and (c) $\mathcal{S}(\overline{v})$ is nonempty by our initial contradiction assumption. □

□

Our next proof shows that Bondareva-Shapley theorem follows from Brouwer's fixed point theorem. Note that it is possible to apply Brouwer's theorem in our first proof to guarantee that the correspondence P has a fixed point by using a continuous selection theorem, as illustrated in Gale and Mas-Colell (1975) and Yannelis and Prabhakar (1983). Instead of following this indirect approach, we construct a continuous "benefit" function in the spirit of the Nash-map of Nash (1951), which allows us to directly apply Brouwer's theorem.

A Second Proof of Bondareva-Shapley Theorem. Towards a contradiction, assume that W has an empty core. It follows that all $v\in\hat{V}$ are blocked by some coalition, and let $\mathcal{S}(v)$ denote the set of coalitions that block v. For each coalition $S\in\mathcal{N}$, define a benefit function $b_S:\hat{V}\to\mathbb{R}_+$ such that $b_S(v)=\max\{W(S)-\sum_{i\in S}v_i,0\}$. Furthermore, define a modified benefit function $\hat{b}_S:\hat{V}\to\mathbb{R}_+$ such that

$$\hat{b}_S(v)=\begin{cases}\min\{b_S(v),\frac{\epsilon}{2}\} & \text{if } |S|=1,\\ b_S(v)\max\{\min_{j\in N}(v_j-\frac{\epsilon}{2}),0\} & \text{if } |S|\geq 2,\end{cases}$$

and define a function $p : \hat{V} \to \hat{V}$ such that, for all $i \in N$

$$p(v)_i = v_i + \frac{\epsilon}{2} \left[\frac{\displaystyle\sum_{S \in \mathcal{N}, i \in S} \hat{b}_S(v) \frac{|N|}{|S|}}{\displaystyle\sum_{S \in \mathcal{N}} \hat{b}_S(v)} - 1 \right]. \tag{2}$$

We show in the Appendix that p is well defined. The function p adapts a construction from Zhou (1994) to our simpler setting and is similar to the Nash-map (1951). The definition of the mapping is a bit intricate, but it has an intuitive interpretation: roughly speaking, given any imputation v, p assigns a higher payoff $p(v)_i > v_i$ to a player i who is a member of blocking coalitions more frequently, and a lower payoff $p(v)_j < v_j$ to a player j who is who is less frequently contained in blocking coalitions.

Lemma 3. *There exists $\overline{v}$ such that $\overline{v} = p(\overline{v})$.*

By construction, p is continuous. Therefore, Lemma 3 follows from an application of Brouwer's fixed point theorem. A fixed point of p is used below to identify a balanced collection of blocking coalitions.

The following lemma, in obvious contradiction with Lemma 3, completes the proof of the Bondareva-Shapley theorem.

Lemma 4. *For all v, $v \neq p(v)$.*

Proof. Suppose, toward a contradiction, that there exists $\overline{v} = p(\overline{v})$. Then, it follows from Equation 2 and $\hat{b}_{S'}(\overline{v}) = 0$ for all non-blocking coalitions S' that the collection $\mathcal{S}(\overline{v})$ is balanced with associated balancing weights

$$\delta_S = \frac{\hat{b}_S(\overline{v}) \frac{|N|}{|S|}}{\sum_{S \in \mathcal{S}(\overline{v})} \hat{b}_S(\overline{v})}$$

for all coalitions $S \in \mathcal{S}(\overline{v})$. It follows that

$$\begin{aligned} \sum_{S \in \mathcal{S}(\overline{v})} \delta_S W(S) &> \sum_{S \in \mathcal{S}(\overline{v})} \delta_S \sum_{i \in S} \overline{v}_i \\ &= \sum_{i \in N} \overline{v}_i \sum_{S \in \mathcal{S}(\overline{v}), i \in S} \delta_S \\ &= W(N), \end{aligned}$$

contradicting the fact that the game is balanced. The inequality follows because $W(S) > \sum_{i \in S} \bar{v}_i$ for all $S \in \mathcal{S}(\bar{v})$ and $\delta_S > 0$ for some $S \in \mathcal{S}(\bar{v})$. □

□

A Appendix

Lemma 5. *For all $v \in \hat{V}$, $p(v) \in \hat{V}$.*

Proof. Pick $v \in \hat{V}$. Note that it follows from

$$\sum_{i \in N} \sum_{S \in \mathcal{N}, i \in S} \hat{b}_S(v) \frac{|N|}{|S|} = \sum_{S \in \mathcal{N}} \frac{\hat{b}_S(v)}{|S|} \sum_{i \in S} |N| = |N| \sum_{S \in \mathcal{N}} \hat{b}_S(v)$$

that $\sum_{i \in N} p(v)_i = \sum_{i \in N} v_i = W(N)$.

It remains to show that $p(v)_i \geq 0$ for all $i \in N$. First, assume that $v_i \leq \epsilon/2$ for some $i \in N$. Then, $\hat{b}_S(v) = 0$ for all $|S| \geq 2$ and $\hat{b}_{\{i\}}(v) > 0$, hence $\sum_{S \in \mathcal{N}} \hat{b}_S(v) > 0$. For all $j \in N$ with $v_j \leq \epsilon/2$, the expression in square brackets in Equation 2 is greater than or equal to

$$\frac{|N|\epsilon/2}{(|N|-1)\epsilon/2} - 1 > 0.$$

Since $\epsilon > 0$, therefore $p(v)_j > v_j \geq 0$. For all $j \in N$ with $v_j > \epsilon/2$, since the expression in square brackets in Equation 2 is greater than or equal to -1, therefore $p(v)_j \geq v_j - \epsilon/2 > 0$.

Second, assume $v_i > \epsilon/2$ for all $i \in N$. It follows from $\mathcal{S}(v) \neq \emptyset$ that there exists some S such that $\hat{b}_S(v) > 0$, hence $\sum_{S \in \mathcal{N}} \hat{b}_S(v) > 0$. Since the expression in square brackets in Equation 2 is always greater than or equal to -1, therefore $p(v)_i > 0$ for all $i \in N$. □

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