Two-Party Competition with Persistent Policies
Supplementary Appendix*

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Abstract

This appendix contains results omitted from 'Two-Party Competition with Persistent Policies'. In particular, it contains the proof of Proposition 1 on long-run outcomes in the absence of policy persistence, as well as the equilibrium constructions that complete the proofs of Propositions 3 and 4 (sufficiency). This appendix also contains all proofs relating to the extensions discussed in the paper: forward-looking voters, limited policy persistence, office-motivated parties, and median uncertainty.

Contents

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A No Policy Persistence

Proof of Proposition 1. As noted in the text, $\frac{1}{1-\delta_J}u_J(M)$ is a subgame perfect equilibrium payoff for party J following any history. Since party J can always enforce this payoff by committing to policy M following any history, this payoff is the lowest subgame perfect equilibrium payoff for J. Hence a policy path $\{y^t\}$ is a subgame perfect equilibrium policy path only if $\sum_{t=0}^{\infty}\delta^t u_J(y^t) \geq \frac{1}{1-\delta_J}u_J(M)$ for all J and all t.

The first step in the proof shows that the game's only subgame perfect equilibrium policy path following any history is the indefinite repetition of the median policy. Strict concavity is needed to ensure that if $y \neq M$ is strictly on party J's side of the median, then $u_J(y) - u_J(M) < u_{-J}(M) - u_{-J}(y)$.[1] This holds since any strictly concave functions u_L and u_R defined on $[0,1]$ with u_L strictly decreasing and u_R strictly increasing can be normalised such that $|u'_L(M)| = |u'_R(M)|$. Suppose $y < M$. By strict concavity, for all $\ell \in [y, M)$ we have $|u'_L(\ell)| < |u'_L(M)| = |u'_R(M)| < |u'_R(\ell)|$, and hence $u_L(y) - u_L(M) < u_R(M) - u_R(y)$.

Consider subgame perfect equilibrium policy path $\{y^t\}$ following some history with $y^0 \neq M$, and suppose that y^0 is on J's side of the median. Define

$$D_J^0 = 0,$$
$$D_{-J}^0 = \frac{u_{-J}(M) - u_{-J}(y^1)}{\delta_{-J}}.$$

For any $i \geq 1$ and y^t (weakly) on J's side of the median, define D_J^t and D_{-J}^t recursively as

$$D_J^t = \max\left\{0, \frac{D_J^{t-1} + [u_J(M) - u_J(y^t)]}{\delta_J}\right\},$$
$$D_{-J}^t = \frac{D_{-J}^{t-1} + [u_{-J}(M) - u_{-J}(y^t)]}{\delta_{-J}}.$$

That is, interpret $D_J^t \geq 0$ as the payoff 'debt' for party J at stage t of subgame perfect equilibrium policy path $\{y^t\}$ relative to path $(M, M, ...)$. This debt collects all deviations from payoff $u_J(M)$; if party J makes a loss with respect to $u_J(M)$ at y^t, then the equilibrium payoff from y^{t+1} needs to yield an excess of at least D_J^t over $\frac{1}{1-\delta_J}u_J(M)$. Debts grow by

[1] Any assumptions that yields this property are sufficient for the result of Proposition 1. For example, if u_L and u_R are weakly concave but strictly concave in a neighbourhood of M.

factor $\frac{1}{\delta_J}$ each period since they are incurred in the current period and reimbursed in later periods. Negative debts are never incurred since party J must be guaranteed the payoff $\frac{1}{1-\delta_J}u_J(M)$ after all histories.

Since $y^0 \neq M$, debts (D_L^0, D_R^0) are such that $D_J^0 > 0$ for some J. Suppose without loss of generality that $\delta_L \leq \delta_R$. First note that for all $t > 0$, it cannot be that $D_L^t = D_R^t = 0$, since $D_J^0 > 0$ and whenever $D_J^t < D_J^{t-1}$, it must be that y^t is strictly on J's side of the median and hence that $D_{-J}^t > D_{-J}^{t-1}$. Next, note that for all J, we have that $\liminf_{i\to\infty} D_J^t = 0$, and also that $D_J^t = 0$ infinitely often. To see this, suppose that there exists some k such that $D_J^t > 0$ for all $t \geq k$. Then the equilibrium value to party J from subgame perfect equilibrium policy path $\{y^t\}_{t=k}^{\infty}$ is strictly less than $\frac{1}{1-\delta_J}u_J(M)$, a contradiction.

Suppose now that $y^0 < M$, and hence that $D_L^0 = 0 < D_R^0$. Then either

i. $D_L^t = 0$ for all $t > 0$.

ii. $D_L^t > 0$ for some $t > 0$.

In case i, it must be that $y^t \leq M$ for all $t > 0$, and hence that $\lim_{t\to\infty} D_R^t \geq \lim_{t\to\infty} \frac{D_R^0}{\delta_R^t} = \infty$, a contradiction. We now see that assuming $y^0 < M$ is without loss of generality. First, any subgame perfect equilibrium policy path that deviates from the median policy after some history must have some subsequence that begins at stage k with debt levels $D_J^k = 0 < D_{-J}^k$. Second, assume instead that $D_L^0 > 0 = D_R^0$. Then either $D_R^t = 0$ for all t, which leads to contradiction, or there exists k such that $D_L^k = 0$, in which case we must have $D_R^k > 0$. Now consider case ii above. There must exist $n > m \geq 0$ with $n - m > 1$ such that $D_R^m > 0$, $D_L^m = D_L^n = 0$ and $D_L^t > 0$ for $t \in \{m+1, ..., n-1\}$. We want to show that $D_R^m < D_R^n$. Consider the sequence $\{\hat{y}^t\}_{t=m+1}^n$ that solves the following minimisation problem.

$$\min_{\{y^t\}_{t=m+1}^n \in X^{n-m}} D_R^n \quad \text{subject to } D_L^m = D_L^n = 0, \text{ given } D_R^m > 0. \tag{1}$$

$\{\hat{y}^t\}_{i=m+1}^n$ exists since D_R^n is continuous and X^{n-m} is compact. Suppose that $\{\hat{y}^t\}_{i=m+1}^n$ is such that $\hat{D}_L^{n-1} > 0$, where $\hat{D}_J^t$ is the debt of party J under $\{\hat{y}^t\}_{i=m+1}^n$. Hence since $D_L^n = 0$ it must be that $\hat{y}^n < M$. Suppose that $\hat{D}_R^{n-2} + [u_R(M) - u_R(\hat{y}^{n-1})] < 0$, which implies that $\hat{D}_R^{n-1} = 0$ and that $\hat{y}^{n-1} > M$. For $\epsilon > 0$, consider $\bar{y}^{n-1} = \hat{y}^{n-1} - \epsilon$ and $\bar{y}^n = \hat{y}^n + \eta_\epsilon$, where η_ϵ is chosen such that $\bar{D}_L^n = 0$. For sufficiently small ϵ, we have that $\bar{D}_R^{n-1} = \hat{D}_R^{n-1} = 0$ and $\bar{D}_R^n < \hat{D}_R^n$, a contradiction. Now suppose that $\hat{D}_R^{n-2} + [u_R(M) - u_R(\hat{y}^{n-1})] \geq 0$. $\hat{D}_R^n$ is strictly increasing in $\hat{y}^{n-1}$ if

$$-\frac{u_R'(\hat{y}^{n-1})}{\delta_R^2} - \frac{u_R'(\hat{y}^n)}{\delta_R}\frac{\mathrm{d}\hat{y}^n}{\mathrm{d}\hat{y}^{n-1}} > 0, \tag{2}$$

where $\frac{\mathrm{d}\hat{y}^n}{\mathrm{d}\hat{y}^{n-1}}$ is given by

$$\frac{u_L'(\hat{y}^{n-1})}{\delta_L^2} - \frac{u_L'(\hat{y}^n)}{\delta_L}\frac{\mathrm{d}\hat{y}^n}{\mathrm{d}\hat{y}^{n-1}} = 0,$$

or $\frac{\mathrm{d}\hat{y}^n}{\mathrm{d}\hat{y}^{n-1}} = -\frac{1}{\delta_L}\frac{u_L'(\hat{y}^{n-1})}{u_L'(\hat{y}^n)}$, which comes from partially differentiating the constraint $D_L^n = 0$ with respect to y^{n-1} and y^n. We can rewrite (2) as

$$\frac{u_L'(\hat{y}^{n-1})}{u_L'(\hat{y}^n)} > \frac{\delta_L}{\delta_R}\frac{u_R'(\hat{y}^{n-1})}{u_R'(\hat{y}^n)}.$$

Say $\hat{y}^{n-1} \geq M$. Then $|u_L'(\hat{y}^{n-1})| \geq |u_R'(\hat{y}^{n-1})|$, $\frac{\delta_L}{\delta_R} \leq 1$ and $|u_L'(\hat{y}^n)| < |u_R'(\hat{y}^n)|$ (since $y^n < M$) imply that (2) holds, and hence that $\{\hat{y}\}_{t=m+1}^n$ does not solve (1), a contradiction. Hence it must be that $\hat{y}^{n-1} < M$.

This pairwise necessary condition for optimality can be used all along the sequence $\{\hat{y}\}_{t=m+1}^n$ to show that a solution to (1) with $\hat{y}^n < M$ must have $\hat{y}^t < M$ for all $t \in \{m+1, ..., n-1\}$. But consider instead sequence $\{\tilde{y}\}_{t=m+1}^n$ with $\tilde{y}^t = M$ for all t. This sequence satisfies the constraints of (1), and is such that $\tilde{D}_R^n = \frac{D_R^m}{\delta_R^{n-m}} < D_R^n$ for any $\{y^t\}_{t=m+1}^n$ with $D^{n-1} < M$. Hence, for the purported equilibrium sequence from above, we have as desired that $D_R^n > D_R^m$. Considering the full policy sequence, we have that whenever $D_L^t > 0$ for $t \in \{m+1, n-1\}$, then $D_R^n > D_R^m$. Furthermore, whenever $D_L^t = 0$ for $t \in \{m+1, n-1\}$, then again $D_R^n > D_R^m$ since $D_L^t = 0$ only if $y^t \leq M$, and as shown above if $D_L^m = 0$, then $D_R^m > 0$. Hence, given the subgame perfect equilibrium path $\{y^t\}$ following some history for which $D_R^k > 0$, we have that $\lim_{t\to\infty} D_R^t = \infty$, a contradiction.

The previous argument shows that the unique subgame perfect equilibrium policy path following any history is $(M, M, ...)$. It remains to be shown that both parties' strategies must call for them to commit to the median following any history. If party J's strategy calls for some policy $y \neq M$ after some history, then party $-J$ must win the election with policy M. Since $y \neq M$, party $-J$ can win the election with a policy it prefers to M, say y'. Since following any deviation, party $-J$ payoffs revert to $\frac{1}{1-\delta_{-J}}{}_{-J}(M)$, deviating to y' is profitable for $-J$. □

B Bounded Extremism: Sufficiency

The following claim completes the proof of Proposition 3: *If $\ell^* \geq 2M - r^*$, the strategy profile $(\sigma_L^{\ell^*}, \sigma_R^{my})$ forms an equilibrium. If $\ell^* < 2M - r^*$, the strategy profile $(\sigma_L^{my}, \sigma_R^{r^*})$ forms an equilibrium.* To show this, suppose that $\ell^* \geq 2M - r^*$. First verify the optimality of L's proposed strategy. Given $(\sigma_L^{\ell^*}, \sigma_R^{my})$ compute

$$V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) = \begin{cases} \frac{1}{1-\delta_L^2} U_L^+(\ell^*) & \text{for } r \in [2M - \ell^*, 1], \\ \frac{1}{1-\delta_L^2} U_L^+(2M - r) & \text{for } r \in [M, 2M - \ell^*), \\ \frac{1}{1-\delta_L^2} u_L(r) & \text{for } r \in [0, M). \end{cases}$$

Note that for all r, r' such that $r > r'$, $\sigma_L(R, r) \in W(R, r)$ and $\sigma_L(R, r) \neq \sigma_L(R, r') \in W(R, r')$,

$$V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) > V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r')).$$

Hence, at any state (R, r) such that $\sigma_L(R, r) \in W(R, r)$, party L cannot profit from one-shot deviation ℓ^d such that $\sigma_L(R, r') = \ell$ for some $r' \neq r$. Hence only one-shot deviations $\ell^d \in [0, \ell^*) \cup (M, 1]$ can be profitable for L at some state.

The value of setting $\ell^d \in [0, \ell^*)$ if winning at (R, r) is

$$\begin{aligned} u_L(\ell^d) + &\delta_L u_L(2M - \ell^d) + \delta_L^2 V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, 2M - \ell^d)) \\ &= U_L^+(\ell^d) + \frac{\delta_L^2}{1 - \delta_L^2} U_L^+(\ell^*). \end{aligned}$$

$\ell^d \in [0, \ell^*)$ is winning only in states (R, r) with $r \in [2M - \ell^d, 1] \cup [0, \ell^d]$. For $r \in [2M - \ell^d, 1]$

$$\begin{aligned} V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) &= \frac{1}{1 - \delta_L^2} U_L^+(l^*) \\ &> U_L^+(\ell^d) + \frac{\delta_L^2}{1 - \delta_L^2} U_L^+(\ell^*), \end{aligned}$$

where the inequality follows from Lemma 1 since $\ell^d < \ell^*$. For $r \in [0, \ell^d]$

$$\begin{aligned} V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) &= \frac{1}{1 - \delta_L^2} u_L(r) \\ &> U_L^+(\ell^d) + \frac{\delta_L^2}{1 - \delta_L^2} U_L^+(\ell^*), \end{aligned}$$

where the inequality follows since $r \leq \ell^d$.

The value of setting $\ell^d \in (M, 1]$ if winning at (R, r) is

$$\frac{1}{1 - \delta_L^2} u_L(\ell^d).$$

$\ell^d \in (M, 1]$ is winning only in states (R, r) with $r \in [2M - \ell^d, M] \cup [\ell^d, 1]$. For $r \in [2M - \ell^d, M]$

$$\begin{aligned} V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) &= \frac{1}{1 - \delta_L^2} u_L(r) \\ &> \frac{1}{1 - \delta_L^2} u_L(\ell^d), \end{aligned}$$

where the inequality follows since $r < \ell^d$. For $r \in [\ell^d, 1]$

$$\begin{aligned} V_L(\sigma_L^{\ell^*}, \sigma_R^{my}; (R, r)) &> \frac{1}{1 - \delta_L^2} u_L(M) \\ &> \frac{1}{1 - \delta_L^2} u_L(\ell^d), \end{aligned}$$

where the first inequality follows since $r > M$, and the second since $\ell^d > M$. Hence, no profitable deviation for L exists and $\sigma_L^{\ell^*}$ is optimal when facing σ_R^{my}.

Now verify the optimality of R's proposed strategy. Given $(\sigma_L^{\ell^*}, \sigma_R^{my})$ compute

$$V_R(\sigma_L^{\ell^*}, \sigma_R^{my}; (L, \ell)) = \begin{cases} u_R(2M - \ell) + \frac{\delta_R}{1-\delta_R^2} U_R^-(\ell^*) & \text{for } \ell \in [0, \ell^*), \\ \frac{1}{1-\delta_R^2} U_R^+(\ell) & \text{for } \ell \in [\ell^*, M), \\ \frac{1}{1-\delta_R^2} u_R(\ell) & \text{for } \ell \in [M, 1). \end{cases}$$

Again, note that for all $\ell < \ell'$, $\sigma_R(L, \ell) \in W(L, \ell)$ and $\sigma_R(L, \ell) \neq \sigma_R(L, \ell') \in W(L, \ell')$

$$V_R(\sigma_L^{\ell^*}, \sigma_R^{my}; (L, \ell)) > V_R(\sigma_L^{\ell^*}, \sigma_R^{my}; (L, \ell')).$$

Hence, at any state (L, ℓ) such that $\sigma_R(L, \ell) \in W(L, \ell)$, party R cannot profit by deviating to any r^d such that $\sigma_R(L, \ell') = r^d$ for some $\ell' \neq \ell$. Hence only one-shot deviations $r^d \in [0, M)$ can be profitable for R at some state. That these cannot be profitable for R follows from a verification similar to that for deviations $\ell^d \in (M, 1]$ for L above. Hence, no profitable deviation for R exists and σ_R^{my} is optimal when facing $\sigma_L^{\ell^*}$.

C Bounded Moderation: Sufficiency

Given a strictly increasing sequence $\{y^t\} \to \hat{\ell}$ with $y^0 = \ell^*$ and y^t, y^{t+1} and y^{t+2} satisfying the conditions of Lemma 4 for all $t \geq 1$, consider the following strategies

$$\sigma^{\hat{\ell}}_{L^*}(R, r) = \begin{cases} \ell^* & \text{for all } r \geq 2M - \ell^*, \\ 2M - r & \text{for all } r \in (2M - y^t, 2M - y^{t-1}) \text{ with } i > 0 \text{ odd}, \\ y^t & \text{for all } r \in [2M - y^t, 2M - y^{t-1}] \text{ with } i > 0 \text{ even}, \\ 2M - r & \text{for all } r \in [M, 2M - \hat{\ell}], \\ Out & \text{for all } r < M. \end{cases}$$

$$\sigma^{\hat{\ell}}_{R}(L, \ell) = \begin{cases} 2M - \ell & \text{for all } \ell < \ell^*, \\ y^t & \text{for all } \ell \in [y^{t-1}, y^t] \text{ with } i > 0 \text{ odd}, \\ 2M - \ell & \text{for all } \ell \in (y^{t-1}, y^t) \text{ with } i > 0 \text{ even}, \\ 2M - \ell & \text{for all } \ell \in [\hat{\ell}, M], \\ Out & \text{for all } \ell > M. \end{cases}$$

If instead $\ell^* < 2M - r^*$, then for robust long-run policy outcome $(\hat{\ell}, 2M - \hat{\ell})$ with $\hat{\ell} > 2M - r^*$, strategies $(\sigma^{\hat{\ell}}_{L}, \sigma^{\hat{\ell}}_{R^*})$ can be constructed in a similar manner with the roles of the parties reversed.

The following claim completes the proof of Proposition 4: *Suppose that $\ell^* \geq 2M - r^*$. Given $\hat{\ell} > \ell^*$ and a strictly increasing sequence $\{y^t\} \to \hat{\ell}$ with $y^0 = \ell^*$ and y^t, y^{t+1} and y^{t+2} satisfying the conditions of Lemma 4 for all $t \geq 1$, strategies $(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_{R})$ form a form a consistent equilibrium under which $\hat{\ell}$ is a robust long-run policy outcome. The equilibrium $(\sigma^{\hat{\ell}}_{L}, \sigma^{\hat{\ell}}_{R^*})$ in the case of $\ell^* < 2M - r^*$ can be determined similarly.* To show this, suppose $\ell^* \geq 2M - r^*$. First verify the optimality of L's proposed strategy. Given $(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_{R})$ and the conditions of the lemma for $\{y^t\}$, compute

$$V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, r)) = \begin{cases} u_L(\ell^*) + \frac{\delta_L}{1-\delta_L^2} U_L^-(y^1) \\ \qquad \text{for } r \in [2M - \ell^*, 1], \\ u_L(2M - r) + \delta_L u_L(2M - y^t) + \frac{\delta_L^2}{1-\delta_L^2} U_L^+(y^{t+1}) \\ \qquad \text{for } r \in (2M - y^t, 2M - y^{t-1}) \text{ with } t > 0 \text{ odd}, \\ u_L(y^t) + \frac{\delta_L}{1-\delta_L^2} U_L^-(y^{t+1}) \\ \qquad \text{for } r \in [2M - y^t, 2M - y^{t-1}] \text{ with } t > 0 \text{ even}, \\ \frac{1}{1-\delta_L^2} U_L^+(2M - r) \text{ for } r \in [M, 2M - \bar{\ell}], \\ \frac{1}{1-\delta_L^2} u_L(r) \text{ for } r \in [0, M). \end{cases}$$

Note that for all r, r' such that $r > r'$, $\sigma_L(R, r) \in W(R, r)$ and $\sigma_L(R, r) \neq \sigma_L(R, r') \in W(R, r')$,

$$V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, r)) > V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, r')).$$

Hence, at any state (R, r) such that $\sigma_L(R, r) \in W(R, r)$, party L cannot profit by deviating to any ℓ^d such that $\sigma_L(R, r') = \ell$ for some $r' \neq r$. Hence only one-shot deviations $\ell^d \in [0, \ell^*) \cup \left(\bigcup_{t>0, t \text{ even}} [y^{t-1}, y^t)\right) \cup (M, 1]$ can be profitable for L at some state. The value to setting $\ell^d \in [0, \ell^*)$ if winning at (R, r) is

$$U_L^+(\ell^d) + \delta_L^2 V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, 2M - \ell^d)) = U_L^+(\ell^d) + \delta_L^2 V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, 2M - \ell^*)).$$

$\ell^d \in [0, \ell^*)$ is winning only in states (R, r) with $r \in [2M - \ell^d, 1] \cup [0, \ell^d]$. For $r \in [2M - \ell^d, 1]$

$$\begin{aligned} V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, r)) &> U_L^+(\ell^d) + \delta_L^2 V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, 2M - \ell^d)) \\ &= U_L^+(\ell^d) + \delta_L^2 V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, 2M - r)). \end{aligned}$$

since

$$\begin{aligned} V_L(\sigma^{\hat{\ell}}_{L^*}, \sigma^{\hat{\ell}}_R; (R, r)) &= u_L(\ell^*) + \frac{\delta_L}{1 - \delta_L^2} U_L^-(y^1) \\ &> \frac{1}{1 - \delta_L^2} U_L^+(\ell^*) \\ &> \frac{1}{1 - \delta_L^2} U_L^+(\ell^d). \end{aligned}$$

The first inequality follows from Lemma 1 and the fact that $y^1 > \ell^*$, and the second inequality from Lemma 1 and the fact that $\ell^d < \ell^*$. That a deviation to $\ell^d \in [0, \ell^*)$ in states (R, r) with $r \in [0, \ell^d]$ is not profitable follows from an argument similar to that in Lemma 4. The value of setting $\ell^d \in [y^{t-1}, y^t)$ for $t > 0$ odd if winning at (R, r) is

$$U_L^+(\ell^d) + \delta_L^2 V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - \ell^d)).$$

$\ell^d \in [y^{t-1}, y^t)$ is winning only in states (R, r) with $r \in [2M - \ell^d, 1] \cup [0, \ell^d]$. Consider

$$\begin{aligned} V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - y^t)) &= \frac{1}{1 - \delta_L^2} U_L^+(y^{t-1}) \\ &= U_L^+(y^{t-1}) + \delta_L^2 V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - y^{t-1})) \\ &\geq U_L^+(\ell^d) + \delta_L^2 V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - \ell^d)), \end{aligned}$$

where the inequality follows from Lemma 1 and the fact that $\ell^* < y^{t-1} \leq \ell^d$ and the fact that $V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - y^{t-1})) = V_L(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}}; (R, 2M - \ell^d))$. Hence, the value to ℓ^d is weakly smaller than the value following action $y^t = \sigma_L(R, 2M - y^t)$, and hence for all states (R, r) with $r \in [2M - \ell^d, 1]$ deviation to ℓ^d by L cannot be profitable. That a deviation to $\ell^d \in [y^{t-1}, y^t)$ in states (R, r) with $r \in [0, \ell^d]$ is not profitable follows from an argument similar to that in the case of equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$, as does the argument that there is no profitable deviation to $\ell^d \in (M, 1]$.

Arguments very similar to those for L above can determine R's payoffs under $(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}})$ and verify that it constitutes an equilibrium. Clearly $\hat{\ell}$ is a robust long-run policy outcome under $(\sigma_{L^*}^{\hat{\ell}}, \sigma_R^{\hat{\ell}})$ since policy dynamics have $\hat{\ell}$ as a limit point starting from all more extreme states.

D Extension: Forward-looking Voters

Proof of Proposition 5. Since I restrict attention to equlibria in which the median voter is decisive, I consider a single representative median voter with utility function u_M and discount factor δ_M. A strategy for the voter is $\sigma_M : (\{L, R\} \times X) \times (X \times \{Out\}) \longrightarrow \{0, 1\}$, where $\sigma_M((I, x), z) = 0$ if and only if the median voter supports incumbent I with policy x in an election opposing it to $-I$ with policy z. Assume that the median voter never abstains so that in particular $\sigma_M((I, x), Out) = 0$ for all (I, x). A *Markov perfect equilibrium with forward-looking voters* is a strategy profile $(\sigma_L, \sigma_R, \sigma_M)$ such that for each state (I, x), (a) given σ_M, (σ_L, σ_R) form a Markov perfect equilibrium, and (b) for any policy z, σ_M is a best-response to (σ_L, σ_R) given $((I, x), z)$.

Consider consistent equilibrium convergence path $\{y^t\}$ with associated consistent equilibrium strategies (σ_L, σ_R). The proof shows that there exist an equilibrium with forward-looking voters $(\sigma'_L, \sigma'_R, \sigma_M)$ that generates the same convergence path. Assume for now that on convergence paths, the median voter votes according to σ_M^{my}. To construct strategies (σ'_L, σ'_R) in the game with forward-looking voters, the profile (σ_L, σ_R) needs to be modified in two ways. First, consider policy y^t such that $\sigma_L(R, 2M - y^t) = y^{t+1}$. For $x \in [y^t, y^{t+1})$, define $z^{t+1}(x) \in [y^t, x)$ such that

i. If

$$u_M(x) - u_M(y^t) > \delta_M \left[V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^t)) - \frac{1}{1 - \delta_M} u_M(x) \right],$$

then $z^{t+1}(x)$ solves

$$u_M(x) - u_M(z^{t+1}(x)) = \delta_M \left[V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^t)) - \frac{1}{1 - \delta_M} u_M(x) \right].$$

ii. If

$$u_M(x) - u_M(y^t) \leq \delta_M \left[V_M(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^t)) - \frac{1}{1 - \delta_M} u_M(x) \right],$$

then $z^{t+1}(x) = y^t$.

That is, R commits to $2M - z^{t+1}(x)$ as 'punishment' for L being in power with policy x as opposed to y^{t+1} and $z^{t+1}(x)$ is the most extreme such punishment that the median voter supports. For y^t such that $\sigma_R(L, y^t) = 2M - y^{t+1}$ and for $x \in (2M - y^{t+1}, 2M - y^t]$, $z^{t+1}(x) \in [y^t, 2M - x)$ can be defined symmetrically.

Second, given some σ_M and $\ell > M$, let $\bar{r}(\ell) > \ell$ be the most extreme commitment by R in state (L, ℓ) that the median voter supports and that R has the incentive to make. If the median voter accepts $\bar{r}(\ell)$, then policy dynamics are 'freed' from the policy traps of equilibria with myopic voters and, after at most one period, the equilibrium path rejoins convergence path $\{y^t\}$. For $r < M$, define $\bar{\ell}(r) < r$ symmetrically. Note that, as with the functions $\{z^{t+1}(\cdot)\}$, $\bar{r}(\cdot)$ and $\bar{\ell}(\cdot)$ are determined only by how parties and the median voter evaluate convergence paths under $(\sigma_L, \sigma_R, \sigma_M^{my})$. Now define strategy σ'_R as

$$\sigma'_L(R, r) = \begin{cases} z^{t+1}(r) & \text{if } r \in (2M - y^{t+1}, 2M - y^t] \text{ for } y^t \text{ such that } \sigma_R(L, y^t) = 2M - y^{t+1}, \\ \bar{\ell}(r) & \text{if } r < M \text{ and } u_L(\bar{\ell}(r)) + \delta_L V_L(\sigma_L, \sigma_R; (L, \bar{\ell}(r))) \geq \frac{1}{1-\delta_L} u_L(r) \\ \sigma_L(R, r) & \text{otherwise.} \end{cases}$$

σ'_R can be defined symmetrically. Let σ_M be a best-response to (σ'_L, σ'_R) in which the median voter supports the opposition party when indifferent. Given the parties' strategies, the median voter has no incentive to vote for the incumbent on a convergence path. Hence, given convergence path policy y^t such that $\sigma_L(R, 2M - y^t) = y^{t+1}$, we have that $V_K(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^t)) = V_K(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^t))$ for $K \in \{L, R, M\}$. I do not describe the median voter's equilibrium strategy explicitly, but instead show how it responds to parties' deviations from the convergence path $\{y^t\}$ to show that parties have no more incentive to deviate from the convergence path under $(\sigma'_L, \sigma'_R, \sigma_M)$ than under $(\sigma_L, \sigma_R, \sigma_M^{my})$.

Consider state (R, r) with $2M - r \in [y^t, y^{t+1})$ for y^t such that $\sigma_R(L, y^t) = 2M - y^{t+1}$. The median voter votes against $\ell \in [y^t, z^{t+1}(r))$ since the payoff to voting in favour of ℓ is

$$u_M(\ell) + \delta_M V_M(\sigma'_L, \sigma'_R, \sigma_M; (L, y^t)) < u_M(r) + \delta_M u_M(z^{t+1}(r)) + \delta_M^2 V_M(\sigma'_L, \sigma'_R, \sigma_M; (L, y^t)),$$

by the definition of $z^{t+1}(r)$, where the right-hand side is the payoff to voting in favour of r. The median voter votes against $\ell \in [y^{t-1}, y^t)$ since the payoff to voting in favour of ℓ is

$$\begin{aligned} &u_M(\ell) + \delta_M u_M(z^i(\ell)) + \delta_M^2 V_M(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^{t-1})) \\ &\qquad < u_M(r) + \delta_M u_M(z^{t+1}(r)) + \delta_M^2 V_M(\sigma'_L, \sigma'_R, \sigma_M; (L, y^t)), \end{aligned}$$

since $|M - \ell| > |M - r|$, $|M - z^i(\ell)| > |M - z^{t+1}(r)|$ and $V_M(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^{t-1})) < V_M(\sigma'_L, \sigma'_R, \sigma_M; (L, y^t))$. Similarly, the median voter votes against $\ell \in [y^{k-1}, y^k)$ for y^k such that $\sigma_L(R, 2M - y^{k-1}) = y^k$ and $k \le t - 2$, and against $\ell \in [y^{k-1}, y^k)$ for y^k such that $\sigma_R(L, y^{k-1}) = 2M - y^k$ and $k \le t - 1$. That is, in state (R, r), the median voter rejects all policies $\ell \in [0, z^{t+1}(r))$. It may or may not vote for policies $\ell \in (z^{t+1}(r), 1]$. A similar argument shows that in state (R, r) with $2M - r \in [y^t, y^{t+1})$ for y^t such that $\sigma_L(R, 2M - y^t) = y^{t+1}$, the median voter rejects any $\ell \in [0, r]$ and may or may not support $\ell \in (r, 1]$, but always supports $\ell = y^{t+1}$.

Now consider parties' incentives. First, whenever a party's equilibrium policy is being accepted, it never gains by committing to policies that are sure to be rejected, since it faces the same choice in the next election. Consider again state (R, r) with $2M - r \in [y^t, y^{t+1})$ for y^t such that $\sigma_R(L, y^t) = 2M - y^{t+1}$. The payoff to party L from policy $\ell \in [z^{t+1}(r), y^{t+1}]$ that is accepted by the median voter is

$$u_L(\ell) + \delta_L u_L(2M - y^{t+1}) + \delta_L^2 V_L(\sigma'_L, \sigma'_R, \sigma_M; (R, 2M - y^{t+1})),$$

which is decreasing in $\ell \in [y^t, y^{t+1})$. From above, policies $\ell \in [0, z^{t+1}(r))$ cannot be profitably proposed since they are rejected by the median voter, while policies in $(y^{t+1}, M]$, if accepted, yield to party L at most the payoff it obtains from such deviations under $(\sigma_L, \sigma_R, \sigma_M^{my})$. Hence committing to $z^{t+1}(r)$ is optimal for party L.

Now consider policy y^t such that $\sigma_L(R, 2M - y^t) = y^{t+1}$ and state (R, r) with $2M - r \in [y^t, y^{t+1})$. The payoff from $\ell \in [2M - r, y^{t+1})$, if accepted by the median voter, is given by

$$\begin{aligned} u_L(\ell) + \delta_L u_L(2M - z^{t+1}(\ell)) + \delta_L^2 V_L(\sigma_L', \sigma_R', \sigma_M; (R, 2M - y^t)) \\ \leq u_L(\ell) + \delta_L u_L(2M - \ell) + \delta_L^2 V_L(\sigma_L', \sigma_R', \sigma_M; (R, 2M - y^t)) \\ < V_L(\sigma_L, \sigma_R, \sigma_M^{my}; (R, 2M - y^t)). \end{aligned}$$

The first inequality follows from $z^{t+1}(\ell) \leq \ell$ and the second since $V_L(\sigma_L', \sigma_R', \sigma_M; (R, 2M - y^t)) > \frac{1}{1-\delta_L^2} U_L^+(\ell)$. This shows that y^{t+1} is L's preferred winning policy in $[y^t, y^{t+1})$ given $(\sigma_L', \sigma_R', \sigma_M)$. As the median voter rejects any policy $\ell \in [0, 2M - r)$, L cannot profitably deviate to such policies. Finally, deviations to any policies $\ell \in (y^{t+1}, M]$ are never profitable since even if they are accepted by the median voter, L's payoffs are no higher than under $(\sigma_L, \sigma_R, \sigma_M^{my})$.

It remains to deal with states (R, r) with $r < M$. By construction, in these states σ_L' is optimal. It needs to be shown that in states (R, r) with $r \geq M$, party L does not want to deviate to some $\ell^d > M$. Consider state (R, r) with $r > M$, and suppose party L deviates to $\ell^d > M$ such that $\sigma_R'(L, \ell^d) = \bar{r}(\ell^d)$ and take $\{y^t\}$ to be the convergence path from $(R, \bar{r}(\ell^d))$. It must be that $y^1 \geq 2M - \bar{r}(\ell^d)$. The payoff to party L from ℓ^d is given by

$$\begin{aligned} u_L(\ell^d) + \delta_L u_L(\bar{r}(\ell^d)) + \sum_{t=1}^{\infty} \delta^{2t}[u_L(y^t) + \delta_L u_L(2M - y^{t+1})] < u_L(\ell^d) + \frac{\delta_L}{1 - \delta_L} u_L(M) \\ < \frac{1}{1 - \delta_L} u_L(M). \end{aligned}$$

The first inequality follows by Lemma 1 and the second since $\ell^d > M$. On the equilibrium path, $V_L(\sigma_L, \sigma_R; (R, r)) \geq \frac{1}{1-\delta_L} u_L(M)$, and hence deviation to ℓ^d is not profitable for L. $\square$

E Extension: Limited Policy Persistence

E.1 Candidate Selection with No Commitment

Consider a model in which parties select policy-motivated candidates to represent them. Specifically, suppose that the leaders of parties L and R have ideal policies 0 and 1, respectively. Given any policy $\hat{x} \in X$, both parties have access to an unlimited pool of potential candidates that have $\hat{x}$ as an ideal policy. Party leaders can select candidates to represent the party, but cannot replace candidates unless they have lost an election. All candidates' policy preferences are publicly observed prior to an election, but candidates cannot commit to implement specific policies. Party leaders and their potential candidates can have different discount factors, which can reflect differences in time horizons between them.

In this model, a state $(I, \hat{x})$ includes the identity of the incumbent party along with the ideal policy of its candidate. A strategy $\hat{\sigma}_J$ for the leaders of party J is now interpreted as a state-contingent choice of candidate. A (Markov) strategy for a candidate of party I with ideal policy $\hat{x}$ is $s_I(\hat{x}) \in X$. Myopic voting in this environment requires that, in all equilibria $(\hat{\sigma}_L, \hat{\sigma}_R, s_L, s_R)$, the opposition party is elected in state $(I, \hat{x})$ if and only if $|M - s_{-I}(\hat{\sigma}_{-I}(I, \hat{x}))| \leq |M - s_I(\hat{x})|$.

Proposition E.1. (σ_L, σ_R) *is a Markov perfect equilibrium of the model with commitment if and only if there exists a Markov perfect equilibrium of the model with candidate selection and no commitment in which parties' selection strategies* $(\hat{\sigma}_L, \hat{\sigma}_R)$ *are such that, for all states* $(I, \hat{x})$*,* $\hat{\sigma}_{-I}(I, \hat{x}) = \sigma_{-I}(I, \hat{x})$*.*

Proof. Consider an equilibrium $(\hat{\sigma}_L, \hat{\sigma}_R, s_L, s_R)$ of the model with candidate selection. Consider a candidate for party I with ideal policy $\hat{x}$ that implements policy x when in office. Since voters' decision to reelect the candidate, as well as the candidate fielded by party $-I$, are conditioned only on the state $(I, \hat{x})$ and not on the policy x, it follows that in any Markov perfect equilibrium, it must be that $s_I(\hat{x}) = \hat{x}$. Hence, given any equilibrium (σ_L, σ_R) of the model with commitment, the profile $(\hat{\sigma}_L, \hat{\sigma}_R, s_L, s_R)$ such that, for all states $(I, \hat{x})$, $\hat{\sigma}_{-I}(I, \hat{x}) = \sigma_{-I}(I, \hat{x})$ and $s_I(\hat{x}) = \hat{x}$, is an equilibrium of the model with candidate selection. Conversely, given any equilibrium $(\hat{\sigma}_L, \hat{\sigma}_R, s_L, s_R)$ of the model with candidate selection, the profile (σ_L, σ_R) such that, for all states (I, x), $\sigma_{-I}(I, x) = \hat{\sigma}_{-I}(I, x)$, is an equilibrium of the model with commitment. □

E.2 Term Limits

Suppose that incumbents can hold office for no more than T terms. In this model, a state (I, x, τ) includes the tenure $\tau \in \{1, ..., T\}$ of the current incumbent.

Proposition E.2. (σ_L, σ_R) *is a Markov perfect equilibrium of the model with no term limit if and only if there exists a Markov perfect equilibrium* (σ_L^T, σ_R^T) *of the model with term limit* $T \geq 2$ *such that, for all states* (I, x)*,* $\sigma_{-I}(I, x, 1) = \sigma_{-I}(I, x)$*.*

Proof. Consider an equilibrium (σ_L^T, σ_R^T) of the model with term limit $T \geq 2$ and any state (I, x, τ). Proposition 1 implies that $\sigma_L^T(I, x, T) = \sigma_R^T(I, x, T) = M$. However, in all states (I, x, τ) such that $\tau < T$, Proposition 2 holds. To see this, consider state (R, r, τ) with $\tau < T$

and $r < M$. For all $\tau < T - 1$, choosing *Out* is optimal for party L, as it obtains a payoff of $u_L(r)$ in all such periods. If $\tau = T - 1$, by choosing *Out* party L obtains a payoff of $u_L(r) + \frac{\delta_L}{1-\delta_L^2} U_L^-(M)$, which is party L's best achievable payoff in that state. Now consider state (R, r, τ) with $\tau < T$ and $r > M$. Party L can still guarantee itself a payoff of at least $\frac{1}{1-\delta_L^2} U_L^+(2M - r)$ by committing to policy $2M - r$. By the previous argument, party L will never commit to a winning policy $\ell > M$. Moreover, party L can benefit from staying *Out* only if $u_L(r) + \frac{\delta_L}{1-\delta_L^2} U_L^+(M) \geq \frac{1}{1-\delta_L^2} U_L^+(2M - r)$, which is false. Hence, nontrivial equilibrium dynamics starting in states (I, x, τ) with $\tau < T$ converge to symmetric alternations. Note that this also implies that if (σ_L^T, σ_R^T) is an equilibrium, then so is $({\sigma'}_L^T, {\sigma'}_R^T)$, which is defined such that

$$
\sigma'_{-I}(I, x, \tau) = \begin{cases} \sigma_{-I}^T(I, x, 1) & \text{if } \tau < T \\ \sigma_{-I}^T(I, x, T) & \text{if } \tau = T. \end{cases}
$$

Consider any equilibrium (σ_L, σ_R) in the model without term limits. In the model with term limit T, define strategies (σ_L^T, σ_R^T) such that

$$
\sigma_{-I}^T(I, x, \tau) = \begin{cases} \sigma_{-I}(I, x) & \text{if } \tau < T \\ M & \text{if } \tau = T. \end{cases}
$$

Note that we have that $\sigma_{-I}(I, x) = \sigma_{-I}^T(I, x, 1)$, and that (σ_L^T, σ_R^T) is an equilibrium since (σ_L, σ_R) is an equilibrium. Now consider any equilibrium (σ_L^T, σ_R^T) in the model with term limits. Then, from above, $({\sigma'}_L^T, {\sigma'}_R^T)$ is also an equilibrium. In the model without term limits, define strategies (σ_L, σ_R) such that $\sigma_{-I}(I, x) = \sigma'_{-I}(I, x, 1)$. Then (σ_L, σ_R) is an equilibrium since (σ_L^T, σ_R^T) is an equilibrium. $\square$

E.3 Costly Policy Adjustments

Proof of Proposition 8. With costly policy adjustment, opposition party $-I$'s strategy is conditioned on state (I, x) while the incumbent I's strategy is conditioned on (I, x, y), where y is the opposition party's policy commitment. Define policy ℓ^c as the solution to

$$
\frac{1}{1 - \delta_L^2} [U_L^+(\max\{\ell^c, \ell^*\}) - U_L^-(\ell^c)] = c, \tag{3}
$$

if it exists, and 0 otherwise. It must be that $\ell^c < M$ since $c > 0$. Furthermore, ℓ^c is decreasing in c, $\lim_{c \to 0} = M$ and there exists $\tilde{c}$ such that $\ell^c = 0$ if and only if $c \geq \tilde{c}$. Policy $r^c \in (M, 1]$ can be defined similarly for party R.

For the remainder of the proof, suppose that $\max\{\ell^*, 2M-r^*\} = \ell^* \leq \ell^c = \max\{\ell^c, 2M-r^c\}$. How to deal with other cases will be easily apparent. To show necessity, first note that the corresponding arguments in the proof of Proposition 3 still hold and that any long-run policy outcome $\ell \leq M$ must be such that $\ell \geq \ell^*$. Suppose now that $\ell \in [\ell^*, \ell^c)$ is a long-run policy outcome. Consider state $(R, 2M-\ell)$. By Proposition 2, the equilbrium payoff to party L in this state is $\frac{1}{1-\delta_L^2}U_L^-(\ell)$. If instead, party L deviates to paying c and adjusting its policy to winning policy ℓ, its payoff is $\frac{1}{1-\delta_L}U_L^+(\ell) - c$. This deviation is profitable since $\ell < \ell^c$, yielding the desired contradiction.

To show sufficiency, consider the strategies $(\sigma_L^c, \sigma_R^{my,c})$ defined as

$$\sigma_L^c(R,r) = \begin{cases} \ell^c & \text{if } r \geq 2M-\ell^c, \\ 2M-r & \text{if } r \in [M, 2M-\ell^c), \\ Out & \text{if } r \in [\min\{r^{cc}, M\}, M), \\ \max\{\ell^c, r\} & \text{if } r < \min\{r^{cc}, M\}. \end{cases}$$

$$\sigma_L^c(L,\ell,r) = \begin{cases} 2M-r & \text{if } r > 2M-\ell^c, \\ Out & \text{if either } r \leq 2M-\ell^c \text{ or } r = Out \text{ and } \ell \leq \ell^{cc}, \\ 0 & \text{if } r = Out \text{ and } \ell > \ell^{cc} \end{cases}$$

$$\sigma_R^c(L,\ell) = \begin{cases} 2M-\ell^c & \text{if } \ell \leq \ell^c, \\ 2M-\ell & \text{if } \ell \in (\ell^c, M], \\ Out & \text{if } \ell \in (M, \max\{\ell^{cc}, M\}], \\ \max\{2M-\ell^c, \ell\} & \text{if } \ell > \max\{\ell^{cc}, M\}. \end{cases}$$

$$\sigma_R^c(R,r,\ell) = \begin{cases} 2M-\ell & \text{if } \ell < \ell^c, \\ Out & \text{if either } \ell \geq \ell^c \text{ or } \ell = Out \text{ and } r \geq r^{cc}, \\ 1 & \text{if } \ell = Out \text{ and } r < r^{cc}, \end{cases}$$

where ℓ^{cc} is defined as the solution to

$$u_L(0)+\delta_L U_L^-(\ell^c)-c = \begin{cases} 1 & \text{if } u_L(0)+\delta_L U_L^-(\ell^c) - c \leq \frac{1}{1-\delta_L}u_L(1), \\ \frac{1}{1-\delta_L}u_L(\ell^{cc}) & \text{if } u_L(0)+\delta_L U_L^-(\ell^c) - c \in (\frac{1}{1-\delta_L}u_L(1), \frac{1}{1-\delta_L}u_L(M)], \\ \frac{1}{1-\delta_L^2}U_L^+(\ell^{cc}) & \text{if } u_L(0)+\delta_L U_L^-(\ell^c) - c \in (\frac{1}{1-\delta_L}u_L(M), \frac{1}{1-\delta_L^2}U_L^+(\ell^c)], \\ u_L(\ell^{cc}) + \frac{1}{1-\delta_L^2}U_L^-(\ell^c) & \text{otherwise.} \end{cases}$$

Define r^{cc} similarly. To simplify the exposition, the strategies have been written in a way that a party's response to action *Out* by an opponent should also be read to describe its response to an opponent choosing a losing policy. Consider the optimality of σ_L^c for party L in state (R, r). Its equilibrium payoff to winning policy $\ell \in [\ell^c, M]$ is $\frac{1}{1-\delta_L^2} U_L^+(\ell)$. Its payoff to winning policy $\ell < \ell^c$ is $u_L(2M - \ell) + \frac{1}{1-\delta_L^2} U_L^+(\ell^c)$, which is strictly less than $\frac{1}{1-\delta_L} u_L(M)$. Its payoff to winning policy $\ell > M$ is $u_L(\ell) + \frac{\delta_L}{1-\delta_L^2} U_L^+(\max\{2M - \ell, \ell^c\})$, which is also strictly less than $\frac{1}{1-\delta_L} u_L(M)$. This verifies the optimality of setting policy $\max\{\ell^c, 2M - r\}$ for those $r \in [0, M]$.

Consider state (R, r) with $r < M$. Party R responds to (R, r, Out) with either a policy of 1 or with *Out*, and *Out* can be a best response for party L only if $\sigma_R^c(R, r, Out) = Out$. When this is the case, the argument that *Out* is optimal for party L is as in the proof for equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$. If instead $\sigma_R^c(R, r, Out) = 1$, the payoff to party L if it stays *Out* is $u_L(1) + \frac{\delta_L}{1-\delta_L} U_L^+(\ell^c)$, which is strictly less than $\frac{1}{1-\delta_L} u_L(M)$. The optimality of the policy prescribed by σ_L^c then follows by the argument of the previous paragraph.

It remains to verify the optimality of σ_L^c in states (L, ℓ, r) for some r. First suppose that $r \neq Out$. If $r > 2M - \ell^c$, then by the arguments from above, if party L decides to pay the adjustment cost it is optimal to commit to policy ℓ^c. Its payoff if it stays *Out* is $u_L(r) + \frac{\delta_L}{1-\delta_L^2} U_L^+(\ell^c) < \frac{\delta_L}{1-\delta_L^2} U_L^-(\ell^c)$. Hence, by the definition of ℓ^c, party L prefers to commit to policy ℓ^c. If $r \leq 2M - \ell^c$, the worst equilibrium payoff for party L if it stays *Out* is $\frac{1}{1-\delta_L^2} U_L^-(\ell^c)$. If instead it pays the adjustment cost, the best payoff it can achieve is, by the arguments from above, $\frac{\delta_L}{1-\delta_L^2} U_L^+(\ell^c) - c$. Hence, by the definition of ℓ^c, party L prefers to stay *Out*.

Now suppose that $r = \{Out\}$. If party L decides to pay the adjustment cost, it will set it preferred policy 0. When it is optimal to do this as opposed to staying *Out* is precisely what is resolved by the definition of ℓ^{cc} above. □

Proof of Corollary 3. The results of the corollary follow from the properties of ℓ^c and r^c. □

F Extension: Office-Motivated Parties

In this section, I allow parties' preferences to display both policy and office-motivation. Suppose that party J's stage payoff to government policy y is the sum of $u_J(y)$ and an office benefit $b > 0$ which party J receives only if it is the party implementing policy y. The model studied so far has $b = 0$. I maintain the assumption that ties are broken in favour of the opposition party. Here, this selection of voting equilibria is not without loss of generality, as the parties have preferences over the pattern of office holding under policy path $\{M, M, ...\}$. However, having ties broken in favour of the opposition party would result in equilibrium if, for example, incumbents' policies were subject to perturbations.

Proposition F.1. *The set of long-run policy outcomes $\mathcal{L}^b$ with office benefit $b > 0$ has the following properties.*

i. $\lim_{b\to 0} \mathcal{L}^b = [\max\{\ell^*, 2M - r^*\}, M]$.

ii. There exits $\bar{b}$ such that $\mathcal{L}^b = [\max\{\ell^*, 2M - r^*\}, M]$ for all $b \geq \bar{b}$.

Proposition F.1 relies on the following result.

Proposition F.2. *Consider the model with office benefits $b > 0$. There exist policies $\ell^{out} \leq \ell^{in} < M$ and $M > r^{in} \geq r^{out}$ such that either*

i. $\ell^{out} < 2M - r^{in}$, $r^{out} > 2M - \ell^{in}$ *and policy $\ell \leq M$ is a long-run policy outcome if and only if $\ell \in [\max\{\ell^*, 2M - \ell^*\}, M]$, or*

ii. $\ell^{out} \geq 2M - r^{in}$ *and there exists policy $\ell^b \in [\max\{\ell^*, 2M - r^*\}, \max\{\max\{\ell^*, 2M - r^*\}, 2M - r^{in}\})$ such that policy $\ell \leq M$ is a long-run policy outcome supported by symmetric alternation only if $\ell \in [\max\{\ell^*, 2M - r^*\}, \ell^b] \cup [\max\{\max\{\ell^*, 2M - r^*\}, \ell^{out}\}, M]$. Furthermore, policy x is a non-trivial long-run policy outcome not supported by symmetric alternation only if $x \in [2M - r^{in}, \ell^{out}]$, or*

iii. $r^{out} \leq 2M - \ell^{in}$, *and the statement is symmetric to ii.*

Policy ℓ^{out} is defined such that if $\ell^{out} > 0$, then party L is indifferent between never holding office and having policy ℓ^{out} implemented forever and gaining office every second election and having policies alternate at $(\ell^{out}, 2M - \ell^{out})$. Policy ℓ^{in} is defined such that if $\ell^{in} > 0$, then party L is indifferent between holding office forever and implementing policy $2M - \ell^{in}$ and holding gaining office every second election and having policies alternate at $(\ell^{in}, 2M - \ell^{in})$. Policies r^{out} and r^{in} can be defined similarly for party R. Hence, if $\ell^{out} \geq 2M - r^{in}$, there is scope for a policy $\ell \in [2M - r^{in}, \ell^{out}]$ to simultaneously give incentives (a) to party R to commit to it knowing that party L will fail to contest all future elections and (b) to opposition party L in state (R, ℓ) not to commit to some winning policy just to gain office. Case i above covers the case in which no such 'bargains' can be sustained. In this case, since parties understand that any attempt to hold office forever will be thwarted, none is made and the set of long-run policy outcomes is as though $b = 0$. Note that cases ii and iii offer only necessary conditions on the sets of long-run policy outcomes with office benefits. Partial converses are derived through equilibrium construction in the proof of Proposition F.1 below.

Proof of Proposition F.2. Define policy $\ell^{out} \in [0, M)$ as the solution to

$$\frac{1}{1-\delta_L} u_L(\ell^{out}) = \frac{1}{1-\delta_L^2}[U_L^+(\max\{\ell^{out}, \ell^*\}) + b], \tag{4}$$

if it exists or as $\ell^{out} = 0$ otherwise. If $\ell^{out} > 0$, then party L is indifferent between never holding office and having policy ℓ^{out} implemented forever and gaining office every second election and having policies alternate at $(\ell^{out}, 2M - \ell^{out})$. Further define policy $\ell^{in} \in [0, M)$ as the solution to

$$\frac{1}{1-\delta_L}[u_L(2M - \ell^{in}) + b] = \frac{1}{1-\delta_L^2}[U_L^+(\max\{\ell^{in}, \ell^*\}) + b] \tag{5}$$

if it exits or as $\ell^{in} = 0$ otherwise. If $\ell^{in} > 0$, then party L is indifferent between holding office forever and implementing policy $2M - \ell^{in}$ and holding gaining office every second election and having policies alternate at $(\ell^{in}, 2M - \ell^{in})$. Policies r^{out} and r^{in} can be defined similarly for party R, where r^* plays the role of ℓ^*. Suppose that $\ell^{out} \in [\ell^*, M)$. Then, (4) yields that $u_L(\ell^{out}) - u_L(2M - \ell^{out}) = \frac{b}{\delta_L}$. Substituting into (5) yields that

$$\begin{aligned} \frac{1}{1-\delta_L} u_L(\ell^{out}) - &\frac{1}{1-\delta_L^2}[U_L^+(\ell^{out}) + b] \\ &= \frac{\delta_L}{1-\delta_L}[u_L(\ell^{out}) - u_L(2M - \ell^{out}) - \delta_L b] \\ &> 0, \end{aligned}$$

and hence $\ell^{in} \in (\ell^{out}, M)$. The same can be shown in the cases in which one or both of ℓ^{out} and ℓ^{in} are smaller than ℓ^*.

Proposition 2, which characterises equilibrium policy paths, no longer obtains if parties care about holding office, since there can be non-trivial long-run policy outcomes in which some party is maintained in office forever.

Proposition F.3. *Consider some equilibrium (σ_L, σ_R) and some state (I, x) along with the policy path $\{y^t\}$ induced by (σ_L, σ_R) starting from (I, x). Then either*

i. $\{y^t\}$ has limit points $(\hat{\ell}, 2M - \ell)$ for some $\ell \leq M$, and both $\sigma_L(R, 2M - \hat{\ell}) = \hat{\ell}$ and $\sigma_R(L, \hat{\ell}) = 2M - \hat{\ell}$, or

ii. $\{y^t\}$ has a unique limit point $x \neq M$, and whenever $x < M$ either $(I^0, y^0) = (R, x)$ or there exists $N > 0$ such that $\sigma_R(L, y^N) = x$. Furthermore, $\sigma_L(R, x) \in \{Out\} \cup [x, 2M - x]^c$. The statement for $x > M$ is symmetric.

Proof. Equilibrium policy path $\{y^t\}$ can have no more than two limit points since, as shown for Proposition 2, all its limit points must be equidistant from the median. First consider part i. By Markov strategies it follows that parties choose winning policies in each period. By Lemma 2, in the limit the payoff to party L can be no less than $\frac{1}{1-\delta_L^2}[U_L^+(\ell)+b]$. As shown for Proposition 2, since $(\hat{\ell}, 2M-\ell)$ are limit points of equilibrium policy sequence $\{y^t\}$, in the limit the payoff to party L can be no more than $\frac{1}{1-\delta_L^2}[U_L^+(\ell)+b]$. Hence, in the limit, party L's payoff is exactly $\frac{1}{1-\delta_L^2}[U_L^+(\ell)+b]$, which implies that, given Markov strategies, $\sigma_L(R, 2M-\hat{\ell}) = \hat{\ell}$ and $\sigma_R(L, \hat{\ell}) = 2M-\hat{\ell}$.

For part ii, suppose that $x < M$ is the unique limit point of $\{y^t\}$. Suppose that $y^t \neq x$ for all t. By Markov strategies, parties must choose winning policies in each period. In the limit, party R's payoff in state (L, y^t) converges to $\frac{u_R(x)}{1-\delta_R} + \frac{b}{1-\delta_R^2}$. Consider a deviation for party R in state (L, y^n) to $2M - y^n > M$ for n sufficiently large. By Lemma 2, party R's payoff would be at least $U_R^+(y^n) + \frac{b}{1-\delta_R^2}$, a contradiction. Hence, there must exist some $N \geq 0$ such that $y^n = x$ for all $n \geq N$. By an argument similar to that above, it must be that for all $n \geq N$, $(I, y^n) = (R, x)$, yielding the rest of part ii. □

Returning to the proof of Proposition F.2, suppose that $\ell^{out} \geq 2M - r^{in}$. I suppose first that $2M - r^{in} \geq \ell^*$ and show that policies in alternation $(\ell, 2M-\ell)$ with $\ell \in (2M - r^{in}, \ell^{out})$ cannot be long-run policy outcomes. Extending the argument to the case in which only $\ell^{out} > \ell^*$ is straightforward. Towards a contradiction, suppose they were. Consider a deviation by party R in state $(R, 2M-\ell)$ to policy ℓ. In state (R, ℓ), the payoff to party L is $\frac{1}{1-\delta_L^2}[U_L^+(\ell)+b]$. Since $\ell < \ell^{out}$, staying *Out* forever yields party L a strictly higher payoff and hence it must be that $\sigma_L(R, \ell) = Out$. Since $\ell > 2M - r^{in}$, then the deviation to ℓ is strictly profitable for party R.

Second, I show that all policies $\ell \notin [2M - r^{in}, \ell^{out}]$ can never be non-trivial long-run policy outcomes. The argument above has shown that such policies are not observed in the long-run as symmetric alternations. By Proposition F.3, if some such policy $\ell > M$ is a non-trivial long-run policy outcome, then there exists an equilibrium (σ_L, σ_R), an initial state $(I, x) \neq (R, \ell)$ and an induced sequence of policies $\{y^t\}$ such that for some $N > 0$ $\sigma_R(L, y^{N-1}) = \ell$ and $\sigma_L(R, \ell) \in \{Out\} \cup [\ell, 2M-\ell]^c$. If $\ell > \ell^{out}$, then *Out* (or any losing policy) is not a best-response for party L in state (R, ℓ). In particular, a deviation to ℓ yields payoff of at least $\frac{1}{1-\delta_L^2}[U_L^+(\ell)+b]$, higher than its equilibrium payoff of $\frac{u_L(\ell)}{1-\delta_L}$ by (4). If $\ell < 2M - r^{in}$, then consider the deviation by R in state (L, y^{N-1}) to policy $2M-\ell$. The payoff to this deviation is at least $\frac{1}{1-\delta_R^2}[U_R^+(\ell)+b]$, higher than its equilibrium payoff of $\frac{1}{1-\delta_R}[u_R(\ell)+b]$ by R's version of (5). A similar argument yields the result for those remaining $\ell < M$.

The final step in the proof is relevant only for cases in which $2M - r^{in} > \ell^{out}$. In that case, some alternations at policies more extreme than $2M - r^{in}$ but within ℓ^* can be ruled out. Consider map $G : [2M - r^{in}, \ell^{out}] \rightarrow [0, 2M - r^{in}]$ defined as the solution to

$$\frac{1}{1-\delta_R}[u_R(\ell) + b] = \frac{1}{1-\delta_R^2}[U_R^+(G(\ell)) + b],$$

if it exists and 0 otherwise. Note that a discontinuity in G can only occur at $G(\ell) = 2M - r^*$ By the definition of r^{in}, we have that $G(2M - r^{in}) = 2M - r^{in}$, $G(\ell) < \ell$ for all $\ell > 2M - r^{in}$ and G is strictly decreasing on $[2M - r^{in}, \ell^{out}]$ when its value is positive. Define mapping $H : [2M - r^{in}, \ell^{out}] \rightarrow [0, \ell^{out}]$ as the solution to

$$\frac{1}{1-\delta_L}u_L(\ell) = \frac{1}{1-\delta_L^2}[U_L^+(H(\ell)) + b],$$

if it exists and 0 otherwise. Note that a discontinuity in H can only occur at $H(\ell) = \ell^*$ By the definition of ℓ^{out}, we have that $H(\ell^{out}) = \ell^{out}$, $H(\ell) < \ell$ for all $\ell < \ell^{out}$ and H is strictly increasing on $[2M - r^{in}, \ell^{out}]$ when its value is positive. Since $G(2M - r^{in}) > H(2M - r^{in})$ and $G(\ell^{out}) > H(\ell^{out})$, if there can exist at most one value $\ell^b \in (\ell^*, 2M - r^{in})$ satisfying $G(\ell^b) = H(\ell^b)$. In all other cases, set $\ell^b = \ell^*$.

For those cases in which $\ell^b > \ell^*$, it remains to be shown that all policies $\ell \in (\ell^b, 2M - r^{in})$ can never be long-run policy outcomes supported by alternation. Consider some long-run policy outcome $\ell \in [\ell^*, 2M - r^{in})$ supported by alternation. By Proposition F.3, it must be that either (i) $\sigma_L(R, \ell) = Out$ for all $\ell \in [2M - r^{in}, \ell^{out}]$, or (ii) $\sigma_L(R, \ell) \in (\ell, M]$ for all $\ell \in [2M - r^{in}, \ell^{out}]$, or (iii) there exists some $\tilde{\ell}$ such that $\sigma_L(R, \tilde{\ell}) = Out$ and for any $\epsilon > 0$, there exists $\hat{\ell}^\epsilon$ such that $\sigma_L(R, \hat{\ell}^\epsilon) \in (\hat{\ell}^\epsilon, M]$ and $|\hat{\ell}^\epsilon - \tilde{\ell}| < \epsilon$. In case (i), consider a deviation by party R in state (L, ℓ) to ℓ^{out}. Party R's payoff from this deviation is $\frac{1}{1-\delta_R}[u_R(\ell^{out}) + b]$, and hence it is not profitable only if $\ell \leq G(\ell^{out}) < H(\ell^{out})$. In case (ii), it must be that $V_L(\sigma_L, \sigma_R; (R, 2M - r^{in})) \geq \frac{1}{1-\delta_L}u_L(2M - r^{in})$. Consider a deviation by party L in state $(R, 2M - \ell)$ to $\sigma_L(R, 2M - r^{in})$. This deviation is not profitable only if $\ell \leq H(2M - r^{in}) < G(2M - r^{in})$. In case (iii), an argument similar to the case (i) above yields that party R cannot profitably deviate to $\tilde{\ell}$ in state (L, ℓ) only if $\ell \leq G(\tilde{\ell})$. Again, an argument similar to the case (ii) above yields that party L cannot profitably deviate to $\sigma(R, \tilde{\ell}^\epsilon)$ for ϵ sufficiently small only if $\ell \leq H(\tilde{\ell})$. Given the properties of functions G and H derived above, it follows that $\min\{G(\tilde{\ell}), H(\tilde{\ell})\} \leq \ell^b$. □

Proof of Proposition F.1. Verifying the claim of Proposition F.1 requires at least a partial answer to sufficiency in Proposition F.2. First, for the case in which $\ell^{out} > 2M - r^{in} > G(\ell^{out}) \geq \ell^* \geq 2M - r^*$, I construct an equilibrium that show that the set of long-run policy outcomes supported by alternation contains the set $[\ell^*, G(\ell^{out})] \cup [\ell^{out}, M]$. Similar constructions apply to other cases. Consider strategies (σ_L^b, σ_R^b) defined as follows.

$$\sigma_L^b(R,r) = \begin{cases} \ell^* & \text{for } r \geq 2M - \ell^* \\ 2M - r & \text{for } r \in [M, 2M-\ell^*) \\ r & \text{for } r \in (\ell^{out}, M] \\ Out & \text{for } r \in [0, G(\ell^{out})] \text{ or } r = \ell^{out} \\ \text{Best of } Out \text{ or } r & \text{otherwise} \end{cases}$$

$$\sigma_R^b(L,\ell) = \begin{cases} 2M - \ell & \text{for } \ell \leq G(\ell^{out}) \\ \ell^{out} & \text{for } \ell \in [G(\ell^{out}), \ell^{out}) \\ 2M - \ell & \text{for } \ell \in [\ell^{out}, M] \\ \ell & \text{for } \ell \in (M, r^{out}] \\ Out & \text{for } \ell \in [\max\{r^{out}, 2M - G(\ell^{out})\}, \max\{r^{out}, 2M - \ell^*\}] \\ \text{Best of } Out \text{ or } \ell & \text{otherwise} \end{cases}$$

Consider the optimality of σ_L^b for party L facing σ_R^b. For states (R,r) with $r \in [M, 2M - \ell^{out}] \cup (2M - G(\ell^{out}), 1]$, the argument follows as in the case of equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$. For states (R,r) with $r \in [2M - \ell^{out}, 2M - G(\ell^{out})]$, the best response of party L must either be $2M - r$ or some policy $\ell \in (\ell^{out}, r]$. Party L's payoff to $2M - r$ is $u_L(2M-r) + b + \frac{\delta_L}{1-\delta_L^2}[U_L^+(\ell^{out}) + b]$, which is higher than $\frac{\delta_L}{1-\delta_L^2}[U_L^+(\ell) + b]$, the payoff to $\ell \in (\ell^{out}, M]$. Since $2M - \ell^{in} < 2M - \ell^{out} \leq r^{in} < r^{out}$, party R responds to any $\ell \in (M, 2M - \ell^{in}]$ with policy ℓ, and hence party L has no incentive to choose such a policy. Similarly, party L has no incentive to choose any policy $\ell \in (2M - \ell^{in}, r]$.

Consider state (R,r) with $r \in [0, G(\ell^{out})]$. The equilibrium payoff to party L if it chooses a winning policy is $\frac{1}{1-\delta_L^2}[U_L^+(r) + b]$ if $r \geq \ell^*$ and $\frac{1}{1-\delta_L^2}[U_L^+(\ell^*) + b]\}$ otherwise. Since $r < \ell^{out}$, staying Out is optimal. Similarly, for state (R,r) with $r \in (\ell^{out}, M]$, the equilibrium payoff to party L is $\frac{1}{1-\delta_L^2}[U_L^+(r) + b]$, and hence policy r is optimal. For those $r \in [G(\ell^{out}), \ell^{out})$, it may be optimal for party L to choose winning policy r even if $r < \ell^{out}$ since its equilibrium payoff to policy r is given by $u_L(r) + b + \frac{\delta_L}{1-\delta_L^2}[U_L^+(G(\ell^{out})) + b]$, which is strictly larger than $\frac{1}{1-\delta_L^2}[U_L^+(r) + b]$. Which of Out or r is optimal is simple, if tedious, to verify. Note that if $r = \ell^{out}$, party L is indifferent between staying Out and choosing winning policy ℓ^{out}, which yields payoff $\frac{\delta_L}{1-\delta_L^2}[U_L^+(\ell^{out}) + b]$.

Now consider the optimality of σ_R^b for party R facing σ_L^b. Again, for states (L,ℓ) with $\ell \in [0, G(\ell^{out})) \cup (\ell^{out}, M]$, the argument follows as in the case of equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$.

For states (L, ℓ) with $\ell \in [G(\ell^{out}), \ell^{out})$, party R's equilibrium payoff is $\frac{1}{1-\delta_R}[u_R(\ell^{out}) + b]$, which since $\ell^{out} > 2M - r^{in}$ is strictly greater than $\frac{1}{1-\delta_R^2}[U_R^+(\ell^{out}) + b]$, the best payoff it can achieve by choosing any winning policy r for which $\sigma_L^b(R, r) \neq Out$. Furthermore, party R's preferred winning policy r for which $\sigma_L^b(R, r) = Out$ is ℓ^{out}, its equilibrium choice.

For those states (L, ℓ) with $\ell \in (M, r^{out}] \cup [\max\{r^{out}, 2M - G(\ell^{out})\}, \max\{r^{out}, 2M - \ell^*\}]$, the argument is similar to that for party L. That is, party R's equilibrium payoff to winning strategy ℓ is $\frac{1}{1-\delta_R^2}[U_R^+(\ell) + b]$ and the definition of policy r^{out} can be applied directly to find which of ℓ or Out is optimal. Again, for those states (L, ℓ) with $\ell > r^{out}$ for which party R's payoff to winning policy ℓ exceeds $\frac{1}{1-\delta_R^2}[U_R^+(\ell) + b]$, a simple verification determines which of ℓ or Out is optimal.

Second, suppose that both $2M - r^{in} \geq \ell^{out}$ and $\ell^{in} \geq 2M - r^{out}$. Then a simple modification of equilibrium $(\sigma_L^{\ell^*}, \sigma_R^{my})$ shows that the bound $\max\{\ell^*, 2M - r^*\}$ on long-run policy outcomes is tight even with office benefits. Consider strategies $(\sigma_L^{\ell^*,b}, \sigma_R^{my,b})$, defined as follows.

$$\sigma_L^{\ell^*,b}(R, r) = \begin{cases} \ell^* & \text{for } r \geq 2M - \ell^* \\ 2M - r & \text{for } r \in [M, 2M - \ell^*) \\ r & \text{for } r \in (\ell^{out}, M] \\ Out & \text{for } r \in [0, \ell^{out}] \end{cases}$$

$$\sigma_R^{my,b}(L, \ell) = \begin{cases} 2M - \ell & \text{for } \ell \in [0, M] \\ \ell & \text{for } \ell \in (M, r^{out}) \\ Out & \text{for } \ell \in [r^{out}, 1] \end{cases}$$

The verification that $(\sigma_L^{\ell^*,b}, \sigma_R^{my,b})$ constitutes an equilibrium mostly follows from the arguments showing that $(\sigma_L^{\ell^*}, \sigma_R^{my})$ constitutes an equilibrium in the absence of office benefits. It remains only to verify that (i) staying Out is optimal for the parties when their strategies call for it and that (ii) no party has an incentive to commit to a policy to which its opponent responds to by staying Out. It is straightforward to see that (i) and (ii) follow from the definitions of (ℓ^{out}, r^{out}) and (ℓ^{in}, r^{in}), respectively.

The two equilibrium constructions from above show that for any $b > 0$, either $[\ell^*, M] = \mathcal{L}^b$ or $\ell^* < \ell^{out}$ and $[\ell^{out}, M] \subseteq \mathcal{L}^b$. Results i and ii Proposition F.1 then follow from the properties of ℓ^{out} (or r^{out} in comparable cases). $\square$

G Extension: Median Uncertainty and Incumbency Advantage

In this section, I assume that there is uncertainty about the location of the median policy which stems from an incumbency bias. I consider the case in which this uncertainty becomes arbitrarily small to investigate the robustness of my results in the model with perfect information. Specifically, fix $\epsilon \in (0, M]$ and consider a state (I, x). In this election, the median policy $M(I, x)$ is such that, with probability q,

$$M(I, x) = M,$$

while with probability $(1 - q)$,

$$M(I, x) = \begin{cases} M + \epsilon & \text{if } x > M + 2\epsilon \\ M - \epsilon & \text{if } x < M - 2\epsilon \\ M & \text{if } x \in [M - 2\epsilon, M + 2\epsilon]. \end{cases}$$

That is, with probability $1 - q$, the median policy is pulled towards the policy championed by the incumbent. Note that in any state (I, x), we have $|M(I, x) - M| \leq |x - M|$. That is, the median policy is never more extreme than the incumbent's policy. Furthermore, opposition parties never need to champion policies on the opposite side of M in order to guarantee that they win an election. For example, in any state (R, r) with $r > M + 2\epsilon$, any policy $\ell \in [2(M + \epsilon) - r, M]$ for party L wins with probability 1, while if $r \in [M, M + 2\epsilon]$, policy M wins with probability 1. The incumbency advantage ensures that, as in previous sections, a long-run policy outcome consists of an alternation at symmetric policies. However, with median uncertainty any such alternation is probabilistic, with the incumbent retaining power with probability $1 - q$.

Proposition G.1. *The set of long-run policy outcomes $\mathcal{L}^\epsilon$ with probabilistic incumbency advantage $\epsilon > 0$ is such that $\lim_{\epsilon \to 0} \mathcal{L}^\epsilon = \{M\}$.*

Proof. A key feature of this parametrisation of incubency advantage is that, as in my main model, sets of winning policies cannot expand over time. Specifically, consider state (I, x), a profile (σ_L, σ_R) and the (random) state path $\{(I^t, x^t)\}_{t=1}^{\infty}$ induced by (σ_L, σ_R) starting from (I, x). Then if policy y wins with probability 0 in state (I, x), it must be that, for all $t \geq 1$, policy y also wins with probability 0 in state (I^t, x^t). To see this, note that, given any t, if $x^t < M - 2\epsilon$, policies in $[x^t, 2(M - \epsilon) - x^t]$ win with probability 1, policies in $(2(M - \epsilon) - x^t, 2M - x^t]$ win with probability q, and all other policies win with probability

0. If instead $x^t \in (M - 2\epsilon, M]$, then policies in $[x^t, 2M - x^t]$ win with probability 1 and all other policies win with probability 0. Similarly, if $x^t > M$, only policies in $[2M - x^t, x^t]$ win with positive probability. Hence, given any $t' > t$, incumbent $I^{t'}$ must champion a policy no more extreme (relative to M) than x^t.

This last fact simplifies the characterisation of long-run policy outcomes when the median location is unknown. Fix any policy $r \geq M$ along with an equilibrium $(\sigma_L^\epsilon, \sigma_R^\epsilon)$. Note that since party L wins with probability 0 in state (R, r) if $\sigma_L^\epsilon(R, r) < 2M - r$, an argument similar to that in Proposition 2 implies that $\sigma_L^\epsilon(R, r) \geq 2M - r$. Similarly, it must be that $\sigma_R^\epsilon(L, 2M - r) \leq r$. Furthermore, we also have that $\sigma_L^\epsilon(R, r) \leq M$. This follows by an argument as in Proposition 2, since for any $\ell > M$, $\sigma_R^\epsilon(L, \ell) = Out$, where this last fact follows since from state (L, ℓ), the feasible policy path preferred by party R is $\{\ell, \ell, ...\}$, which it can achieve only by staying *Out*. Hence, again by an argument as in Proposition 2, policy $r \geq M$ is a (non-trivial) long-run policy outcome under $(\sigma_L^\epsilon, \sigma_R^\epsilon)$ if and only if $\sigma_L^\epsilon(R, r) = 2M - r$ and $\sigma_R^\epsilon(L, 2M - r) = r$.

For the remainder of the proof, assume that policy $r \geq M + 2\epsilon$ is a long-run policy outcome under $(\sigma_L^\epsilon, \sigma_R^\epsilon)$. In that case, since

$$V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (R, r)) = qu_\ell(2M - r) + (1 - q)u_\ell(r) + \delta_L\Big[qV_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (L, 2M - r)) + (1 - q)V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (R, r))\Big],$$

and

$$V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (L, 2M - r)) = (1 - q)u_\ell(2M - r) + qu_\ell(r) + \delta_L\Big[(1 - q)V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (L, 2M - r)) + qV_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (R, r))\Big],$$

it can be computed that

$$V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (R, r)) = \frac{1}{[1 - \delta_L(1 - q)]^2 - \delta_L^2 q^2}\left[qu_L(2M - r) + [(1 - q)(1 - \delta_L) + \delta_L q]u_L(r)\right].$$

Consider an alternative strategy σ_L^d for party L which is such that

$$\sigma_L^d(R, r) = \begin{cases} 2(M + \epsilon) - r & \text{if } r \geq M + 2\epsilon \\ M & \text{if } r \in [M, M + 2\epsilon) \\ Out & \text{if } r < M, \end{cases}$$

and let $\{x^t\}_{t=1}^\infty$ be the (random) policy path induced by σ_L^d starting from (R, r). Further, recursively define a sequence of policies $\{\tilde{x}^t\}_{t=1}^{\infty}$ such that

$$\tilde{x}^1 = 2(M+\epsilon) - r,$$
$$\tilde{x}^t = \sigma_R^{\epsilon}(L, x^{t-1}) \quad \text{for } t>1 \text{ even},$$
$$\tilde{x}^t = \begin{cases} 2(M+\epsilon) - x^{t-1} & \text{if } x^{t-1} \geq M+2\epsilon \\ M & \text{if } x^{t-1} \in [M, M+2\epsilon). \end{cases} \quad \text{for } t>1 \text{ odd.}$$

Note that, by construction, $\tilde{x}^t \leq M$ for all $t > 1$ odd. Hence, it must be that, for any $t > 1$ even, we have that $\tilde{x}^t \geq M$, since $\sigma_R^{\epsilon}(L,\ell) \geq M$ for all $\ell \leq M$. The sequence $\{\tilde{x}^t\}$ corresponds to the policy path induced by the profile $(\sigma_L^d, \sigma_R^{\epsilon})$ from (R, r) that would result if party R won with probability 1 whenever it implemented an equilibrium response to a policy of party L under σ_L^d in state (L,ℓ) with $\ell \leq M$. Let $V_L^{d,\epsilon}$ be the payoff to party L from the profle $(\sigma_L^d, \sigma_R^{\epsilon})$ in state (R, r). We have that

$$\begin{aligned} V_L^{d,\epsilon} &= u_L(2(M+\epsilon)-r) + \mathbb{E}\sum_{t>1} \delta_L^{t-1}(x^t) \\ &\geq u_L(2(M+\epsilon)-r) + \sum_{t>1,\text{ t even}} \delta_L^{t-1}\left[u_L(\tilde{x}^t) + \delta_L u_L(\tilde{x}^{t+1})\right] \\ &= u_L(2(M+\epsilon)-r) + \sum_{t>1,\text{ t even}} \delta_L^{t-1}\left[u_L(\tilde{x}^t) + \delta_L u_L(\min\{2(M+\epsilon)-\tilde{x}^t, M\})\right] \\ &\geq \frac{1}{1-\delta_L^2}\left[u_L(2(M+\epsilon)-r) + \delta_L u_L(r)\right], \end{aligned}$$

where the first inequality follows since, for t even, $\tilde{x}^t \geq \tilde{x}^{t-1}$, and the final inequality follows by the concavity of u_L since, for all t even, $M \leq \tilde{x}^t \leq r$. Finally, since

$$\begin{aligned} \lim_{\epsilon\to 0}\left[V_L(\sigma_L^{\epsilon}, \sigma_R^{\epsilon}; (R,r)) - V_L^{d,\epsilon}\right] &\leq V_L(\sigma_L^{\epsilon}, \sigma_R^{\epsilon}; (R,r)) - \frac{1}{1-\delta_L^2}[u_L(2M-r) + \delta_L u_L(r)] \\ &= \frac{(1-q)(1-\delta_L)^2}{(1-\delta_L^2)\left[[1-\delta_L(1-q)]^2 - \delta_L^2 q^2\right]}\left[u_L(r) - u_L(2M-r)\right] \\ &< 0, \end{aligned}$$

this implies that $r \geq M+2\epsilon$ is a long run policy outcome under $(\sigma_L^{\epsilon}, \sigma_R^{\epsilon})$ only if ϵ is sufficiently large, establishing the claim of Proposition G.1. □

The next result relies on equilibrium construction, which is challenging with median uncertainty. To simplify the arguments involved, I assume that, as in Section 5.3, parties have office benefit $b > 0$. Further, I assume that b is large, so that in equilibrium, opposition parties always commit to policies that win with probability 1 (such a policy is always available, e.g., committing to the incumbent's policy). This allows a simple resolution of parties' trade-off between policy and winning probability.

Proposition G.2. *Consider the model with uncertain incumbency advantage $\epsilon > 0$ and suppose that both parties have large office benefit. Then given any policies $\hat{\ell} \in (\max\{\ell^*, 2M - r^*\}, M]$ and $r^0 > 2M - \max\{\ell^*, 2M - r^*\}$, and any $T > 0$, there exist $\hat{\epsilon} > 0$ and an equilibrium $(\sigma_L^{\hat{\epsilon}}, \sigma_R^{\hat{\epsilon}})$ such that, if $\{x^t\}$ is the sequence of policies induced by $(\sigma_L^{\hat{\epsilon}}, \sigma_R^{\hat{\epsilon}})$ starting from (R, r^0), then, for all $t \leq T$, $|M - x^t| \geq M - \hat{\ell}$.*

In words, given any policy $\hat{\ell}$ more moderate than the most extreme long-run policy outcome from the model with perfect information, level of uncertainty about the median can be chosen small enough to ensure that there exists an equilibrium under which, starting from a policy more extreme than $\hat{\ell}$, it takes an arbitrarily long time for policy dynamics to become more moderate than $\hat{\ell}$.

Proof of Proposition G.2. Given $\ell \in [0, M]$, define the sequence of policies $\{\tilde{x}_\ell^t\}_{t=0}^\infty$ recursively such that

$$\tilde{x}_\ell^0 = \ell$$

$$\tilde{x}_\ell^t = \begin{cases} 2(M-\epsilon) - \tilde{x}_\ell^{t-1} & \text{if } \tilde{x}_\ell^{t-1} \in [0, M-2\epsilon) \\ 2M - \tilde{x}_\ell^{t-1} & \text{if } \tilde{x}_\ell^{t-1} \in [M-2\epsilon, M] \end{cases} \quad \text{if } t > 0 \text{ is odd}$$

$$\tilde{x}_\ell^t = \begin{cases} 2(M+\epsilon) - \tilde{x}_\ell^{t-1} & \text{if } \tilde{x}_\ell^{t-1} \in (M+2\epsilon, 1] \\ 2M - \tilde{x}_\ell^{t-1} & \text{if } \tilde{x}_\ell^{t-1} \in [M, M+2\epsilon] \end{cases} \quad \text{if } t > 0 \text{ is even}$$

Note that, given any $\epsilon > 0$, the sequence of policies $\{\tilde{x}_\ell^t\}$ corresponding to $\ell \in [0, M]$ reaches policy x such that $|M - x| \leq 2\epsilon$ for the first time at some finite $T_\ell^\epsilon \geq 0$, with $|M - \tilde{x}_\ell^t| = |M - \tilde{x}_\ell^{T_\ell^\epsilon}|$ for all $t \geq T_\ell^\epsilon$. Also, if T_ℓ^ϵ is even, we have that $\tilde{x}_\ell^{T_\ell^\epsilon} \leq M$, while if T_ℓ^ϵ is odd, we have that $\tilde{x}_\ell^{T_\ell^\epsilon} \geq M$ Given $\ell \in [0, M]$, let $\tilde{U}_L(\ell)$ be the payoff to party L from the sequence of policies $\{\tilde{x}_\ell^t\}$. That is,

$$\tilde{U}_L(\ell) = \sum_{t=0}^\infty \delta^t u_L(\tilde{x}_\ell^t)$$

$$= \begin{cases} \sum_{t=0,\, t \text{ even}}^{T_\ell^\epsilon - 2} \delta^t \left[u_L(\tilde{x}_\ell^t) + \delta_L u_L(2(M-\epsilon) - \tilde{x}_\ell^t)\right] + \frac{\delta_L^{T_\ell^\epsilon}}{1-\delta_L^2}\left[u_L(\tilde{x}_\ell^{T_\ell^\epsilon}) + \delta_L u_L(2M - \tilde{x}_\ell^{T_\ell^\epsilon})\right] \\ \qquad \text{if } T_\ell^\epsilon \geq 2 \text{ is even} \\ \sum_{t=0,\, t \text{ even}}^{T_\ell^\epsilon - 1} \delta^t \left[u_L(\tilde{x}_\ell^t) + \delta_L u_L(2(M-\epsilon) - \tilde{x}_\ell^t)\right] + \frac{\delta_L^{T_\ell^\epsilon + 1}}{1-\delta_L^2}\left[u_L(2M - \tilde{x}_\ell^{T_\ell^\epsilon}) + \delta_L u_L(\tilde{x}_\ell^{T_\ell^\epsilon})\right] \\ \qquad \text{if } T_\ell^\epsilon \geq 1 \text{ is odd.} \end{cases}$$

Given that $\tilde{U}_L$ is continuous on $[0, M]$, let $\ell^{*,\epsilon} = \max\{\arg\max_{\ell \in [0,M]} \tilde{U}(\ell)\}$ be its most moderate maximiser. Note that $\lim_{\epsilon \to 0} \ell^{*,\epsilon} = \ell^*$. Similarly, we can define $r^{*,\epsilon} \in [M, 1]$. Suppose that $\ell^{*,\epsilon} \geq 2M - r^{*,\epsilon}$ (a similar argument applies in the opposite case) and define strategy profile $(\sigma_L^\epsilon, \sigma_R^\epsilon)$ such that

$$\sigma_L^\epsilon(R, r) = \begin{cases} \ell^{*,\epsilon} & \text{if } r \in [0, \ell^{*,\epsilon}] \cup [2(M+\epsilon) - \ell^{*,\epsilon}, 1] \\ 2(M+\epsilon) - r & \text{if } r \in [M + 2\epsilon, 2(M+\epsilon) - \ell^{*,\epsilon}) \\ 2M - r & \text{if } r \in [M, M + 2\epsilon) \\ r & \text{if } r \in (\ell^{*,\epsilon}, M), \end{cases}$$

$$\sigma_R^\epsilon(L, \ell) = \begin{cases} 2(M-\epsilon) - \ell & \text{if } \ell \in [0, M - 2\epsilon] \\ 2M - \ell & \text{if } \ell \in (M - 2\epsilon, M] \\ \ell & \text{if } \ell \in (M, 1]. \end{cases}$$

This strategy profile is meant to mimic the key properties of the profile $(\sigma_L^*, \sigma_R^{my})$ from Proposition 3, but adjusted to incumbency advantage and high office benefit: party L takes policy dynamics to its preferred 'ϵ-alternation' if possible, and initiates the most extreme such alternation that wins with probability 1 when not possible.

Consider state (R, r). If ϵ is small enough that $\ell^{*,\epsilon} < M - 2\epsilon$, it follows from arguments as in Proposition 3 that σ_L^ϵ is optimal for party L against σ_R^ϵ if $r \in [M, M + 2\epsilon]$. Similarly, since office benefits are high, party L always champions a policy that wins with probability 1, and it can be verified that σ_L^ϵ is optimal for party L against σ_R^ϵ if $r \in [0, M)$. Now suppose that $r \in [M + 2\epsilon, 1]$, and note that $V_L(\sigma_L^\epsilon, \sigma_R^\epsilon; (R, r)) = \tilde{U}_L(2(M+\epsilon) - r)$. Since office benefits are high and only policies $\ell \in [2(M+\epsilon) - r, r]$ allow party L to win with probability 1, championing policy $\ell \leq 2(M+\epsilon) - r$ is never optimal for party L. Since no policy $\ell \in (M, r]$ is optimal for party L, under profile $(\sigma_L^\epsilon, \sigma_R^\epsilon)$, only policies $\ell \in [2(M+\epsilon) - r, M]$ can be optimal for party L. Hence, a verification argument very similar to that of Proposition 3 establishes that $(\sigma_L^\epsilon, \sigma_R^\epsilon)$ is an equilibrium as long as $\tilde{U}_L$ is decreasing on $[\ell^{*,\epsilon}, M - 2\epsilon]$. While it is intuitive that $\tilde{U}_L$ should be single-peaked around $\ell^{*,\epsilon}$, establishing this is not trivial, in particular since $\tilde{U}_L$ is not everywhere differentiable. The key property that I will exploit is that non-differentiabilities in $\tilde{U}_L(\ell)$ are driven by terms in the tail of sequence $\{\tilde{x}_\ell^t\}$, so that they can safely be ignored when ϵ is small.

Let $\ell^{m,\epsilon}$ be such that

$$\ell^{m,\epsilon} = \underset{\ell \in [0, M-2\epsilon]}{\arg\max} \left[u_L(\ell) + \delta_L u_L(2(M-\epsilon) - \ell) \right].$$

Note that $\ell^{m,\epsilon}$ is well-defined since u_L is strictly concave, which also implies that, given any $\ell \in (\ell^{m,\epsilon}, M-2\epsilon]$, we have that

$$u_L(\ell) + \delta_L u_L(2(M-\epsilon)-\ell) < u_L(\ell^{m,\epsilon}) + \delta_L u_L(2(M-\epsilon)-\ell^{m,\epsilon}),$$

and hence that $\ell^{*,\epsilon} \leq \ell^{m,\epsilon}$. Also, it must be that $\tilde{U}_L$ is decreasing on $[\ell^{m,\epsilon}, M-2\epsilon]$, so it only needs to be established that $\tilde{U}_L$ is decreasing on $[\ell^{*,\epsilon}, \ell^{m,\epsilon}]$. Finally, note that $\ell^{m,\epsilon} < \ell^* < M$, and that $\lim_{\epsilon \to 0} \ell^{m,\epsilon} = \ell^*$.

For fixed $\epsilon > 0$, T_ℓ^ϵ is a piecewise constant, decreasing function of $\ell \in [0, M-2\epsilon]$, and $\tilde{U}_L^\epsilon$ is differentiable at all ℓ at which T_ℓ^ϵ is continuous. If T_ℓ^ϵ is not continuous at $\tilde{\ell}$, then $\lim_{\ell \nearrow \tilde{\ell}} T_\ell^\epsilon = T_{\tilde{\ell}}^\epsilon + 1$. Furthermore, note that if $T_{\tilde{\ell}}^\epsilon$ is odd, it must be that $\tilde{x}_\ell^{T_\ell^\epsilon - 1}$ converges to $M+2\epsilon$ from above as $\ell \nearrow \tilde{\ell}$, while if $T_{\tilde{\ell}}^\epsilon$ is even, $\tilde{x}_\ell^{T_\ell^\epsilon - 1}$ converges to $M-2\epsilon$ from below as $\ell \nearrow \tilde{\ell}$. It follows that

$$\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) = \begin{cases} \frac{\partial}{\partial^- \ell}\tilde{U}_L(\tilde{\ell}) - \frac{\delta_L^{T_{\tilde{\ell}}^\epsilon + 1}}{1-\delta_L^2}\left[u_L'(M)(1+\delta_L) - [u_L'(M-2\epsilon) + \delta_L u_L'(M+2\epsilon)]\right] & \text{if } T_{\tilde{\ell}}^\epsilon \text{ is odd} \\ \frac{\partial}{\partial^- \ell}\tilde{U}_L(\tilde{\ell}) - \frac{\delta_L^{T_{\tilde{\ell}}^\epsilon + 1}}{1-\delta_L^2}\left[u_L'(M)(1+\delta_L) - [u_L'(M+2\epsilon) + \delta_L u_L'(M-2\epsilon)]\right] & \text{if } T_{\tilde{\ell}}^\epsilon \text{ is even.} \end{cases}$$

Hence, by the strict concavity of u_L, we have that $\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) < \frac{\partial}{\partial^- \ell}\tilde{U}_L(\tilde{\ell})$ if $T_{\tilde{\ell}}^\epsilon$ is even, while, if ϵ is small enough that $\ell^{*,\epsilon} < M-2\epsilon$, we have that $\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) > \frac{\partial}{\partial^- \ell}\tilde{U}_L(\tilde{\ell})$ if $T_{\tilde{\ell}}^\epsilon$ is odd. Note that there must exist $\tilde{\ell} > \ell^{*,\epsilon}$ such that, for all $\ell \in [\ell^{*,\epsilon}, \tilde{\ell}]$, $\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\ell) < 0$, and let $\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) = \eta < 0$. To see this, first suppose that T_ℓ^ϵ is continuous at $\ell^{*,\epsilon}$, so that $\frac{\partial}{\partial \ell}\tilde{U}_L(\ell^{*,\epsilon}) = 0$ and, since u_L is strictly concave, $\frac{\partial^2}{\partial \ell^2}\tilde{U}_L(\ell^{*,\epsilon}) < 0$. Second, suppose that T_ℓ^ϵ is not continuous at $\ell^{*,\epsilon}$, so that $\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\ell^{*,\epsilon}) \leq 0$. But then it also follows from the strict concavity of u_L and the piecewise continuity of T_ℓ^ϵ that $\frac{\partial^2}{\partial^+ \ell^2}\tilde{U}_L(\ell^{*,\epsilon}) < 0$, again yielding the desired result.

For fixed ℓ, T_ℓ^ϵ is a decreasing function of ϵ. Fix $\tilde{\epsilon}$ such that, for all $\epsilon \leq \tilde{\epsilon}$,

$$\frac{\delta_L^{T_{\ell^{m,\epsilon}}^\epsilon}}{(1-\delta_L)(1-\delta_L^2)}\left[u_L'(M-2\epsilon) + \delta_L u_L'(M+2\epsilon) - u_L'(M)(1+\delta_L)\right] < -\eta,$$

and consider any $\ell \in [\tilde{\ell}, \ell^{m,\epsilon}]$. First, suppose that $T_{\tilde{\ell}}^\epsilon = T_\ell^\epsilon$. Then from above we have that

$$\begin{aligned} \frac{\partial}{\partial^+ \ell}\tilde{U}_L(\ell) &< \frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) \\ &= \eta < 0. \end{aligned}$$

Second, suppose that $T^\epsilon_{\tilde{\ell}} - T^\epsilon_\ell \geq 1$. Then

$$\begin{aligned}
\frac{\partial}{\partial^+ \ell}\tilde{U}_L(\ell) &\leq \frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) + \delta_L^{T^\epsilon_\ell} \sum_{t=0}^{T^\epsilon_{\tilde{\ell}} - T^\epsilon_\ell - 1} \frac{\delta_L^t}{1-\delta_L^2}\left[u_L'(M-2\epsilon) + \delta_L u_L'(M+2\epsilon) - u_L'(M)(1+\delta_L)\right] \\
&\leq \frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) + \frac{\delta_L^{T^\epsilon_{\ell^m,\epsilon}}}{(1-\delta_L)(1-\delta_L^2)}\left[u_L'(M-2\epsilon) + \delta_L u_L'(M+2\epsilon) - u_L'(M)(1+\delta_L)\right] \\
&< \frac{\partial}{\partial^+ \ell}\tilde{U}_L(\tilde{\ell}) - \eta \\
&= 0,
\end{aligned}$$

where the second inequality follows since $T^\epsilon_\ell \geq T^\epsilon_{\ell^m,\epsilon}$. This establishes the claim that, for $\epsilon \leq \tilde{\epsilon}$, $\tilde{U}_L$ is decreasing on $[\ell^{*,\epsilon}, M-2\epsilon]$.

To complete the proof of Proposition G.2, suppose that $\ell^* \geq 2M - r^*$ and fix any policies $r^0 \in (2M-\ell^*, 1]$ and $\hat{\ell} \in (\ell^*, M]$ as well as $T > 0$. Note that under $(\sigma^\epsilon_L, \sigma^\epsilon_R)$, we have that $\sigma^\epsilon_L(R, r^0) = \ell^{*,\epsilon}$. Since $\lim_{\epsilon\to 0} \ell^{*,\epsilon} = \ell^*$ and $\lim_{\epsilon\to 0} T^\epsilon_{\ell^{*},\epsilon} = \infty$, there exists $\hat{\epsilon}$ such that, for all $\epsilon \leq \hat{\epsilon}$, $|x^t_{\ell^*,\epsilon} - M| < M - \hat{\ell}$ for all $t \leq T$, as desired.

□