Chapter 3

Inter-temporal Economics

In this chapter we will re-label the basic model in order to focus the analysis on the question of inter-temporal choice. This will reveal information about borrowing and savings behaviour and the relationship between interest rates and the time-preferences of consumers. In particular, we will see that interest rates are just another way to express a price ratio between time dated commodities.

3.1 The consumer’s problem

In what follows, we will consider the simple case of just two commodities, consumption today — denoted $c_1$ — and consumption tomorrow — $c_2$. These take the place of the two commodities, say apples and oranges, in the standard model of the previous chapter. Since it is somewhat meaningless to speak of income in a two period model, indeed, part of our job here is to find out how to deal with and evaluate income streams, we will employ an endowment model. The consumer is assumed to have an endowment of the consumption good in each period, which we will denote by $m_1$ and $m_2$, respectively. The consumer is assumed to have preferences over consumption in both periods, represented by the utility function $u(c_1, c_2)$. 
3.1.1 Deriving the budget set

Our first job will be to determine how the consumer’s budget may be expressed in this setting. That, of course, will depend on the technology for storing the good and on what markets exist. Note first of all that it is never possible to consume something before you have it: the consumer will not be able to consume in period 1 any of the endowment in period 2. In contrast, it may be possible to store the good, so that quantities of the period 1 endowment not consumed in period 1 are available for consumption in period 2. Of course, such storage may be subject to losses (depreciation). Ideally, the consumer has some investment technology (such as planting seeds) which allows the consumer to “invest” (denoting that the good is used up, but not in utility generating consumption) period 1 units in order to create period 2 units. Finally, the consumer may be able to access markets, which allow lending and borrowing. We consider these in turn.

No Storage, No Investment, No Markets

If there is no storage and no investment, then anything not consumed today will be lost forever and consumption cannot be postponed to tomorrow. You may want to think of this as 100% depreciation of any stored quantity. No markets means that there is also no way to trade with somebody else in order to either move consumption forward or backward in time. The consumer finds himself therefore in the economically trivial case where he is forced to consume precisely the endowment bundle. The budget set then is just that one point: $B = \{c \in \mathbb{R}_+^2 | c_i = m_i, i = 1, 2\}$.

Storage, No Investment, No Markets

This is a slightly more interesting case where consumption can be postponed. Since there are no markets, no borrowing against future endowments is possible. Storage is usually not perfect. Let $\delta \in (0, 1)$ denote the rate of depreciation — for example, our consumption good may spoil, so that the outside layers of the meat will not be edible in the next period, or pests may eat some of the stuff while it is in storage (a serious problem in many countries.) A typical budget set then is

$$B = \{(c_1, c_2) \mid c_2 \leq m_2 + (1 - \delta)(m_1 - c_1), 0 \leq c_1 \leq m_1\}. $$
The quantity \((m_1 - c_1)\) in this expression is the amount of period 1 endowment not consumed in period 1, usually called savings, and denoted \(S\). We can therefore express the consumer’s utility maximization problem in three equivalent ways:

\[
\begin{align*}
\max_{c_1, c_2 \in B} & \quad u(c) \\
\max_{c_1 \leq m_1} & \quad u(c_1, m_2 + (1 - \delta)(m_1 - c_1)) \\
\max_{S \leq m_1} & \quad u(m_1 - S, m_2 + (1 - \delta)S)
\end{align*}
\]

In the second case only the level of consumption in period 1, \(c_1\), is a choice variable. It implies the consumption in period 2. The third line simply re-labels that same maximization and has savings in period 1 — whatever is not consumed — as the choice variable. All of these will give the same answer, of course! The left diagram in Figure 3.1 gives the diagrammatic representation of this optimization problem (with two different (!) preferences indicated by representative indifference curves.)

One thing to be careful about is the requirement that \(0 \leq c_1 \leq m_1\). We could employ Kuhn-Tucker conditions to deal with this constraint, but usually it suffices to check after the fact. For example, a consumer with CD preferences \(u(c_1, c_2) = c_1 c_2\) faced with a price ratio of unity would like to consume where \(c_1 = c_2\) (You ought to verify this!) However, if the endowment happens to be \((m_1, m_2) = (5, 25)\) then this is clearly impossible. We therefore conclude that there is a corner solution and consumption occurs at the endowment point \(m\).

**Storage, Investment, No Markets**

How is the above case changed if investment is possible? Here I am thinking of physical investment, such as planting, not “market investment”, as in lending. Suppose then that the consumer has the same storage possibility as previously, but also has access to an investment technology. We will model this technology just as if it were a firm, and specify a function that gives the returns for any investment level: \(y = f(x)\). This function must either have constant returns to scale or decreasing returns to scale for things to work easily (and for the second order conditions of the utility maximization problem to hold.) Suppose the technology exhibits CRS. In that case the marginal return of investment, \(f'(x)\), is constant. It either is larger or smaller than the return of storage, which is \(1 - \delta\). Since the consumer will want to maximize the budget set, he will choose whichever has the higher return, so if \(f'(x) > (1 - \delta)\) the consumer will invest, and he will store otherwise. The
budget then is

\[ B = \begin{cases} 
  \{ (c_1, c_2) \mid c_2 \leq m_2 + f(m_1 - c_1), 0 \leq c_1 \leq m_1 \} & \text{if } (1 - \delta) < f'() \\
  \{ (c_1, c_2) \mid c_2 \leq m_2 + (1-\delta)(m_1 - c_1), 0 \leq c_1 \leq m_1 \} & \text{otherwise} 
\end{cases} \]

What if the investment technology exhibits decreasing returns to scale? Suppose the most interesting case, where \( f'(x) > (1 - \delta) \) for low \( x \), but the opposite is true for high \( x \). Suppose further that not only the marginal return falls, but that the case is one where \( f(m_1) < (1 - \delta)m_1 \), so that the total return will be below that of storage if all first period endowment is invested. What will be the budget set? Clearly (?) any initial amounts not consumed in period 1 should be invested, not stored. But when should the consumer stop investing? The maximal investment \( \bar{x} \) is defined by \( f'(\bar{x}) = (1 - \delta) \). Any additional unit of foregone period 1 consumption should be stored, since the marginal return of storage now exceeds that of further investment. The budget then is

\[ B = \begin{cases} 
  \{ (c_1, c_2) \mid c_2 \leq m_2 + f(m_1 - c_1), m_1 - \bar{x} \leq c_1 \leq m_1 \} \\
  \{ (c_1, c_2) \mid c_2 \leq m_2 + f(\bar{x}) + (1-\delta)(m_1 - \bar{x} - c_1), 0 \leq c_1 \leq m_1 - \bar{x} \}
\end{cases} \]

where \( \bar{x} \) is defined by \( (1 - \delta) = f'(\bar{x}) \)

Storage, No Investment, Full Markets

We now allow the consumer to store consumption between periods 1 and 2, with some depreciation, and to trade consumption on markets. Instead of the usual prices, which are an expression of the exchange ratio of, say good 2 for good 1, we normally express things in terms of interest rates when we deal with time. Of course, one can always convert between the two without much trouble if some care is taken. The key idea is that a loan of \( P \) dollars today will pay back the principal \( P \) plus some interest income. At an interest rate \( r \), this is an additional \( rP \) dollars. Thus, 1 dollar today “buys” \((1 + r)\) dollars tomorrow. Put differently, the price of 1 of today’s dollars is \((1 + r)\) of tomorrow’s. Similarly, for a payment of \( F \) dollars tomorrow, how many dollars would somebody be willing to pay? \( F/(1 + r) \), of course, since \( F/(1 + r) + rF/(1 + r) = F \). We therefore have an equivalence between \( P \) dollars today and \( F \) dollars tomorrow, provided that \( P(1 + r) = F \), or \( P = F/(1 + r) \). By convention the value \( P \) is called the present value of \( F \), while \( F \) is the future value of \( P \).

Given these conventions, let us now derive the budget set of a consumer who may borrow and lend, but has no (physical) investment technology.
Storage is not an option in order to move consumption from period 2 to period 1. The consumer may, however, forgo some future consumption for current consumption by purchasing the appropriate borrowing contract. Based on what we have said above, if he is willing to pay, in period 2, an amount of \((m_2 - c_2) > 0\), then in period 1 he can receive at most \((m_2 - c_2)/(1 + r)\) units. Thus one constraint in the budget is \(m_1 \leq c_1 \leq m_1 + (m_2 - c_2)/(1 + r)\). On the other hand, consumption can be postponed in two ways: storage and lending. If the consumer stores the good his constraint on period 2 consumption is as derived previously, \(c_2 \leq m_2 + \delta(m_1 - c_1)\). If he uses the market instead, his constraint becomes \(c_2 \leq m_2 + (m_1 - c_1)(1 + r)\). As long as \(\delta\) and \(r\) are both strictly positive the second of these is a strictly larger set than the first. Since consumption is a good (more is better) the consumer will not use storage, and the effective constraint on future consumption will be \(c_2 \leq m_2 + (m_1 - c_1)(1 + r)\). This is indicated in the right diagram in Figure 3.1, where there are two “budget lines” for increasing future consumption. The lower of the two is the one corresponding to storage, the higher the one corresponding to a positive interest rate on loans.

Manipulation of the two constraints shows that they are really identical. Indeed, the consumer’s budget can be expressed as any of

\[
B = \{(c_1, c_2) \mid c_2 \leq m_2 + (1 + r)(m_1 - c_1), \ c_1, c_2 \geq 0\},
\]

\[
= \{(c_1, c_2) \mid c_1 \leq m_1 + \frac{(m_2 - c_2)}{1 + r}, \ c_1, c_2 \geq 0\},
\]

\[
= \{(c_1, c_2) \mid (m_2 - c_2) + (1 + r)(m_1 - c_1) \geq 0, \ c_1, c_2 \geq 0\},
\]

\[
= \{(c_1, c_2) \mid (m_1 - c_1) + \frac{(m_2 - c_2)}{1 + r} \geq 0, \ c_1, c_2 \geq 0\}.
\]

The first and third of these are expressed in future values, the second and fourth in present values. It does not matter which you choose as long as you adopt one perspective and stick to it! (If you recall, we are free to pick a numeraire commodity. Here this means we can choose a period and denominate everything in future- or present-value equivalents for this period. This, by the way, is the one key “trick” in doing inter-temporal economics: pick one time period as your “base”, convert everything into this time period, and then stick to it! Finally note that we are interested in the budget line, generally. Replacing inequalities with equalities in the above you note that all are just different ways of writing the equation of a straight line through the endowment point \((m_1, m_2)\) with a slope of \((1 + r)\).
Storage, Investment, Full Markets

What if everything is possible? As above, storage will normally not figure into the problem, so we will ignore it for a moment. How do (physical) investment and market investment interact in determining the budget? As in the case of storage and investment, the first thing to realize is that the (physical) investment technology will be used to the point where its marginal (gross rate of) return \( f'(\cdot) \) is equal to that of putting the marginal dollar into the financial market. That is, the optimal investment is determined by \( f'(\bar{x}) = (1 + r) \). The second key point is that with perfect capital markets it is possible to borrow money to invest. The budget thus is bounded by a straight line through the point \((m_1 - \bar{x}, m_2 + f(\bar{x}))\) with slope \((1 + r)\).

### 3.1.2 Utility maximization

After the budget has been derived the consumer’s problem is solved in the usual fashion, and all previous equations which characterize equilibrium apply. Suppose the case of markets in which an interest rate of \( r \) is charged. Then it is necessary that

\[
\frac{u_1(c)}{u_2(c)} = \frac{(1 + r)}{1}.
\]

What is the interpretation of this equation? Well, on the right hand side we have the slope of the budget line, which would usually be the price ratio \( p_1/p_2 \). That is, the RHS gives the cost of future consumption in terms of current consumption (recall that in terms of units of goods we have \((1/p_2)/(1/p_1)\)). On the left hand side we have the ratio of marginal utilities, that is, the MRS. It tells us the consumer’s willingness to trade off current consumption for future consumption. This will naturally depend on the consumer’s time preferences.
Let us take a concrete example. In macro-economics we often use a so-called **time-separable utility function** like \( u(c_1, c_2) = \ln c_1 + \beta \ln c_2; \beta < 1 \). This specification says that consumption is ranked equally within each period (the same sub-utility function applies within each period) but that future utility is not as valuable as today’s and hence is discounted. One of the key features of such a separable specification is that the marginal utilities of today’s consumption and tomorrow’s are independent. That is, if I consider \( \partial u(c_1, c_2)/\partial c_i \), I find that it is only a function of \( c_i \) and not of \( c_j \). This feature makes life much easier if the goal is to solve some particular model or make some predictions. Note that the **discount factor** \( \beta \) is related to the consumer’s **discount rate** \( \rho \): \( \beta = 1/(1 + \rho) \). For this specific function, the equation characterizing the consumer’s optimum then becomes

\[
\frac{c_2}{\beta c_1} = (1 + r) \Rightarrow \frac{c_2}{c_1} = \beta(1 + r) \Rightarrow \frac{c_2}{c_1} = \frac{1 + r}{1 + \rho}.
\]

Some interesting observations follow from this. First of all, if the private discount rate is identical to the market interest rate, \( \rho = r \), the consumer would prefer to engage in **perfect consumption smoothing**. The ideal consumption path has the consumer consume the same quantity each period. If the consumer is more impatient than the market, so that \( \rho > r \), then the consumer will favour current consumption, while a patient consumer for whom \( \rho < r \) will postpone consumption. This is actually true in general with additively separable functions, since \( u'(c_1) = u'(c_2) \) iff \( c_1 = c_2 \).

In order to achieve the preferred consumption path the consumer will have to engage in the market. He will either be a **lender** \( (c_1 < m_1) \) or a **borrower** \( (c_1 > m_1) \). This, of course, depends on the desired consumption mix compared to the consumption mix in the consumer’s endowment.

Finally, we can do the usual **comparative statics**. How will a change in relative prices affect the consumer’s wellbeing and consumption choices? The usual facts from revealed preference theory apply: If the interest rate increases (i.e., current consumption becomes more expensive relative to future consumption) then

- A lender will remain a lender.
- A borrower will borrow less (assuming normal goods) and may switch to being a lender.
- A lender will become better off.
- A borrower who remains a borrower will become worse off.
A borrower who switches to become a lender may be worse or better off.

These can be easily verified by drawing the appropriate diagram and observing the restrictions implied by the weak axiom of revealed preference.

We can also see some of these implications by considering the Slutsky equation for this case. In this case the demand function for period 1 consumption is \( c_1(p_1, p_2, M) \), where \( M = p_1 m_1 + p_2 m_2 \). It follows from the chain rule that

\[
\frac{dc_1(\cdot)}{dp_1} = \frac{\partial c_1(\cdot)}{\partial p_1} + \frac{\partial c_1(\cdot)}{\partial M} \frac{\partial M}{\partial p_1},
\]

but \( \frac{\partial M}{\partial p_1} = m_1 \). The Slutsky equation tells us that

\[
\frac{\partial c_1(\cdot)}{\partial p_1} = \frac{\partial h_1(\cdot)}{\partial p_1} - c_1(\cdot) \frac{\partial c_1(\cdot)}{\partial M}
\]

and hence we obtain

\[
\frac{dc_1(\cdot)}{dp_1} = \frac{\partial h_1(\cdot)}{\partial p_1} - (c_1(\cdot) - m_1) \frac{\partial c_1(\cdot)}{\partial M}.
\]

This equation is easily remembered since it is really just the Slutsky equation as usual, where the weighting of the income effect is by market purchases only. In the standard model without endowment, all good 1 consumption is purchased, and hence subject to the income effect. With an endowment, only the amount traded on markets is subject to the income effect.

Now to the analysis of this equation. We know that the substitution effect is negative. We also know that for a normal good the income effect is positive. The sign of the whole expression therefore depends on the term in brackets, in other words on the lender/borrower position of the consumer. A borrower has a positive bracketed term. Thus the whole expression is certainly negative and a borrower will consume less if the price of current consumption goes up. A lender will have a negative bracketed term, which cancels the negative sign in the expression, and we know that the total effect is less negative than the substitution effect. In fact, the (positive) income effect could be larger than the (negative) substitution effect and current consumption could go up!
3.2 Real Interest Rates

So far we have used the usual economic notion of prices as exchange rates of goods. In reality, prices are not denoted in terms of some numeraire commodity, but in terms of money (which is not a good.) This may lead to the phenomenon that there is a price ratio of goods to money, and that this may change over time, an effect called inflation if money prices of goods go up (and deflation otherwise.) We can modify our model for this case by fixing the price of the numeraire at only one point in time. We then can account for inflation/deflation. Doing so will require a differentiation between nominal and real interest rates, however, because you get paid back the principal and interest on an investment in units of money which has a different value (in terms of real goods) compared to the one you started out with. So, let $p_1 = 1$ be the (money) price of the consumption good in period 1 and let $p_2$ be the (money) price of the consumption good in period 2. Let the nominal interest rate be $r$, which means that you will get interest of $r$ units of money per unit. The budget constraint for the two period problem then becomes:

$$B :: p_2c_2 = p_2m_2 + (1 + r)(m_1 - c_1).$$

In other words, the total monetary value of second period consumption can at most be the monetary value of second period endowment plus the monetary value of foregone first period consumption.

We can now ask two questions: what is the rate of inflation and what is the real interest rate.

The rate of inflation is the rate $\pi$ such that $(1 + \pi)p_1 = p_2$, i.e., $\pi = (p_2 - p_1)/p_1$. The real interest rate can now be derived by rewriting the above budget to have the usual look:

$$c_2 = m_2 + \frac{1 + r}{p_2}(m_1 - c_1) = m_2 + \frac{1 + r}{1 + \pi}(m_1 - c_1) = m_2 + (1 + \hat{r})(m_1 - c_1).$$

Thus, the real interest rate is $\hat{r} = (r - \pi)/(1 + \pi)$, and as a rule of thumb (for small $\pi$) it is common to simply use $\hat{r} \sim (r - \pi)$. Note however that this exaggerates the real interest rate.

3.3 Risk-free Assets

Another application of this methodology is for a first look at assets. It is easiest if we first look at financial assets only. A financial asset is really just
a promise to pay (sometimes called an IOU, from the phrase “I owe you”.) If we assume that the promise is known to be kept, that the amount it will pay back is known, and that the value of what it pays back when it does is known, then we have a risk-free asset. While there are few such things, a government bond comes fairly close. Note that assets do not have to be financial instruments: they can also be real, such as a dishwasher, car, or plant (of either kind, actually.) By calling those things assets we focus on the fact that they have a future value.

3.3.1 Bonds

A bond is a financial instrument issued by governments or large corporations. Bonds are, as far as their issuer is concerned, a loan. A bond has a so called face value, which is what it promises to pay the owner of the bond at the maturity date $T$ of the the bond. A normal bond also has a a sequence of coupons, which used to be literally coupons that were attached to the bond and could be cut off and exchanged for money at indicated dates. If we denote by $F$ the face value of the bond, and by $C$ the value of each coupon, we can now compute the coupon rate of the bond. For simplicity, assume a yearly coupon for now (we will see more financial math later which allows conversion of compounding rates into simple rates). In that case the coupon rate is simply $\frac{C}{F} 100\%$. A strip bond is a bond which had all its coupons removed ("stripped"). The coupons themselves then generate a simple annuity — a fixed yearly payment for a specified number of years — which could be sold separately. Another special kind of bond is a consol, which is a bond which pays its coupon rate forever, but never pays back any face value.

We can now ask what the price of such a bond should be today. Since the bond bestows on its owner the right to receive specified payments on specified future dates, its value is the current value of all these future payments. Future payments are, of course, converted into their present values by using the appropriate interest rate:

$$PV = \frac{1}{(1 + r)} C + \frac{1}{(1 + r)^2} C + \ldots + \frac{1}{(1 + r)^T} C + \frac{1}{(1 + r)^T} F.$$

Denote the coupon rate by $c$. Then we can simplify the above equation to yield

$$PV = F \left( \frac{1}{(1 + r)^T} + c \sum_{i=1}^{T} \frac{1}{(1 + r)^i} \right).$$
From this equation follows one of the more important facts concerning the price of bonds and the interest rate. Recall that once a bond is issued $T$, $F$, and $C$ are fixed. That means that the present price of the bond needs to adjust as the interest rate $r$ varies. Since $r$ appears in the denominator, as the interest rate rises the present value of a bond falls. Bond prices and interest rates are **inversely related**. Is the present price of a bond higher or lower than the maturity value? In the latter case the bond is trading at a discount, in the former at a premium. This will depend on the relationship between the interest rate and the coupon rate. Intuitively, if the coupon rate exceeds the interest rate the bond is more valuable, and thus its price will be higher.

### 3.3.2 More on Rates of Return

You may happen to own an asset which, as mentioned previously, is the right to a certain stream of income or benefits (services). This asset happens to also have a current market price (we will see later where that might come from and what conditions it will have to satisfy.) The question you now have is, what is this asset’s rate of return?

Let us start with the simple most case of no income/services, but an asset that only has a current and a future price, $p_0$ and $p_1$, respectively. Your rate of return then is the money you gain (or loose) as a percentage of the cost of the asset, in other words the **per dollar rate of return** is $\frac{p_1 - p_0}{p_0}$. We can now ask what conditions such a rate of return might have to satisfy in equilibrium. Assume, therefore, a case where there is complete certainty over all assets’ returns, i.e., future prices. All consumers would want to buy that asset which has the highest rate of return, since that would move their individual budget line out the most. In equilibrium, if a consumer is willing to hold more than one asset, then it must be true that both assets have the same rate of return. Call this rate $r$. Then we know that $r = \frac{p_1 - p_0}{p_0} = \frac{p_1}{p_0} - 1$, or $(1 + r) = \frac{p_1}{p_0}$. This condition must hold for all assets which are held, otherwise there would be arbitrage opportunities. It is therefore also known as a **zero arbitrage condition**. Recall that arbitrage refers to the activity of selling high while buying low in order to make a profit. For example, assume that there were an asset for which $p_1/p_0 > 1 + r$. Then what I should do is to borrow money at the interest rate $r$, say $I$ dollars, and use those funds to buy the good in question, i.e., purchase $I/p_0$ units. Then wait and sell the goods in the next period. That will yield $Ip_1/p_0$ dollars. I also have to pay back my loan, at
$(1 + r)I$ dollars, and thus I have a profit of $p_1/p_0 - (1 + r)$ per dollar of loan. Note that the optimal loan size would be infinite. However, the resulting large demand would certainly drive up current prices (while also lowering future prices, since everybody expects a flood of the stuff tomorrow), and this serves to reduce the profitability of the exercise. In a zero-arbitrage equilibrium we therefore must have $(1 + r) = p_1/p_0$, or, more tellingly, $p_0 = \frac{p_1}{1 + r}$. The correct market price for an asset is its discounted future value!

This discussion has an application to the debate about pricing during supply or demand shocks. For example, gasoline prices during the Gulf war, or the alleged price-gouging in the ice-storm areas of Quebec and Ontario: What should the price of an item be which is already in stock? Many people argue that it is unfair to charge a higher price for in-stock items. Only the replacement items, procured at higher cost, should be sold at the higher cost. While this may be “ethical” according to some, it is easily demonstrated to violate the above rule: The price of the good tomorrow will be determined by demand and supply tomorrow, and apparently all are agreed that that price might well be higher due to a large shift out in the demand and/or reduction in supply. Currently I own that good, and have therefore the choice of selling it tomorrow or selling it today. I would want to obtain the appropriate rate of return on the asset, which has to be equal between the two options. Thus I am only willing to part with it now if I am offered a higher price which foreshadows tomorrow’s higher price. Should I be forced not to do so I am forced to give money away against my will and better judgment. This would normally be considered unethical by most (just try and force them to give you money.)

Of course, assets are not usually all the same, and we will see this later when we introduce uncertainty. For example, a house worth $100,000 and $100,000 cash are not equivalent, since the cash is immediately usable, while the house may take a while to sell — it is less “liquid.” The same is true for thinly traded stocks. Such assets may carry a liquidity premium — an illiquidity punishment, really — and will have a higher rate of return in order to compensate for the potential costs and problems in unloading them. This can, of course, be treated in terms of risk, since the realization of the house’s value is a random variable, at least in time, if not in the amount. Of course, there are other kinds of risk as well, and in general the future price of the asset is not known. (Note that bonds are an exception to some degree. If you choose to hold the bond all the way to the maturity date you do know the precise stream of payments. If you sell early, you face the uncertain sale price which depends on the interest rate at that point in time.)
Assets may also yield consumption returns while you hold them: a car or house are examples, as are dividend payments of stocks or interest payments of bonds. For one period this is still simple to deal with: The asset will generate benefit (say rent saved, or train tickets saved) of $b$ and we thus compute the rate of return as $\frac{p_1 - p_0 + b}{p_0}$. If the consumer holds multiple assets in equilibrium, then we again require that this be equal to the rate of return on other assets. Complicating things in the real world is the fact that assets often differ in their tax treatment. For example, if the house is a principal residence any capital gains (the tax man’s term for $p_1 - p_0$, and to add insult to injury they ignore inflation) are tax free. For another asset, say a painting, this is not true. Equilibrium requires, of course, that the rates of return as perceived by the consumer are equalized, and thus we may have to use an after tax rate for one asset and set it equal to an untaxed rate for another.

### 3.3.3 Resource Depletion

The simple discounting rules above can also be applied to gain some first insights into resource economics. We can analyse the question of simple resource depletion: at what rate should we use up a non-renewable resource. We can also analyse when a tree (or forest) should be cut down.

Assume a non-renewable resource currently available at quantity $S$. For simplicity, first assume a fixed annual demand $D$. It follows that there are $\frac{S}{D}$ years left, after which we assume that an alternative has to be used which costs $C$. Thus the price in the last year should be $p_{\frac{S}{D}} = C$. Arbitrage implies that $p_{t+1} = (1 + r)p_t$, so that $p_0 = C/(1 + r)^{\frac{S}{D}}$. Note that additional discoveries of supplies lower the price since they increase the time to depletion, as do reductions in demand. Lowering the price of the alternative also lowers the current price. Finally, increases in the discount rate lower the price.

This approach has a major flaw, however. It assumes demand and supply to be independent of price. So instead, let us assume some current price $p_0$ as a starting value and let us focus on supply. When will the owner of the resource be willing to sell? If the market rate of return on other assets is $r$ then the resource, which is just another asset, will also have to generate that rate of return. Therefore $p_1 = (1 + r)p_0$, and in general we’d have to expect $p_t = (1 + r)^tp_0$. Note that the price of the resource is therefore increasing with time, which, in general equilibrium, means two things: demand will
fall as customers switch to alternatives, and substitutes will become more competitive. Furthermore we might expect more substitutes to be developed. We will ultimately run out of the resource, but it is nearly always wrong to simply use a linear projection of current use patterns. This fact has been established over and over with various natural resources such as oil, tin, copper, titanium, etc.

What about renewable resources? Consider first the ‘European’ model of privately owned land for timber production as an example. Here we have a company who owns an asset — a forest — which it intends to manage in order to maximize the present value of current and future profits. When should it harvest the trees? Each year there is the decision to harvest the tree or not. If it is cut it generates revenue right away. If it continues to grow it will not generate this revenue but instead generate more revenue tomorrow (since it is growing and there will be more timber tomorrow.) It follows that the two rates of return should be equalized, that is, the tree should be cut once its growth rate divided by its current size has slowed to the market interest rate. This fact has a few implications for forestry: Faster growing trees are a better investment, and thus we see mostly fast growing species replanted, instead of, say, oaks, which grow only slowly. (This discussion is *ceteris paribus* — ignoring general equilibrium effects.) Furthermore, what if you don’t own the trees? What if you are the James Bond of forestry, with a (time-limited) license to kill? In that case you will simply cut the trees down either immediately or before the end of your license, depending on the growth rate. Of course, in Canada most licenses are for mature forests, which nearly by definition have slow or no growth — thus the thing to do is to clear cut and get out of there. The Europeans, critical of clear-cutting, forget that they have long ago cut nearly all of their mature forests and are now in a harvesting model with mostly high growth forests.

As a final note, notice that lack of ownership will also impact the replanting decision. As we will see later in the course, if we treat the logger as an agent of the state, the state has serious incentive problems to overcome within this principal agent framework.

### 3.3.4 A Short Digression into Financial Economics

I thought it might be useful to provide you with a short refresher or introduction to multi-period present value and compound interest computations. For starters, assume you put $1 in the bank at 5% interest, computed yearly, and that all interest income is also reinvested at this 5% rate. How much
money will you have in each of the following years? The answer is

$$1.05, 1.05^2, 1.05^3, \ldots 1.05^t.$$ 

The important fact about this is that a simple interest rate and a compounded interest rate are not the same, since with compounding there is interest on interest. For example, if you get a loan at 12%, it matters how often this is compounded. Let us assume it is just simple interest; You then owe $1.12 for every dollar you borrowed at the end of one year. What you will quickly find out is that banks don’t normally do that. They at least compound semi-annually, and normally monthly. Monthly compounding would mean that you will owe $(1 + \frac{12}{12})^{12} = 1.1268$. On a million dollar loan this would be a difference of $6825.03$. In other words, you are really paying not a 12% interest rate but a 12.6825% simple interest rate. It is therefore very important to be sure to know what interest rate applies and how compounding is applied (semi-annual, monthly, etc.?)

Here is a handy little device used in many circles: the rule of 72, sometimes also referred to as the rule of 69. It is used to find out how long it will take to double your money at any given interest rate. The idea is that it will approximately take $72/r$ periods to double your money at an interest rate of $r$ percent. The proof is simple: we want to solve for the $t$ for which

$$\left(1 + \frac{r\%}{100}\right)^t = 2 \Rightarrow t \ln\left(1 + \frac{r\%}{100}\right) = \ln 2.$$ 

However, for small $x$ we know that $\ln(1 + x) \sim x$, thus

$$\frac{t \cdot \frac{r\%}{100}}{100} \sim \ln 2 \Rightarrow t \sim \frac{100 \ln 2}{r\%} = \frac{69.3147}{r\%}$$

but of course 72 has more divisors and is much easier to work with.

The power of compounding also comes into play with mortgages or other installment loans. A mortgage is a promise to pay every period for a specified length (typically 25 years, i.e., 300 months) a certain payment $p$. This is also known as a simple annuity. What is the value of such a promise, i.e., its present value? We need to compute the value of the following sum:

$$\delta p + \delta^2 p + \delta^3 p + \ldots + \delta^n p.$$ 

Here $\delta = 1/(1+r)$, where $r$ is the interest rate we use per period. Thus

$$\text{PV} = \delta (p + \delta p + \delta^2 p + \ldots + \delta^{n-1} p)$$

$$= \delta p \left(1 + \delta + \delta^2 + \ldots + \delta^{n-1}\right)$$
\[
PV = \frac{\delta p (1 - \delta^n)}{1 - \delta} = \frac{1 - (1 + r)^{-n}}{r} p
\]

(Recall in the above derivation that for \( \delta < 1 \) we have \( \sum_{i=1}^{\infty} \delta^i = 1/(1 - \delta) \).)

The above equation relates four variables: the principal amount, the payment amount, the payment periods, and the interest rate. If you fix any three this allows you to derive the fourth after only a little bit of manipulation. A final note: In Canada a mortgage can be at most compounded semi-annually. Thus the effective interest rate per month is derived by solving \((1 + r/2)^2 = (1 + r_m)^{12}\). If you are quoted a 12\% interest rate per year the monthly rate is therefore \((1.06)^{1/6} - 1 = 0.975879418\%\). The effective yearly interest rate in turn is \((1.06)^2 - 1 = 12.36\%\), and by law the bank is supposed to tell you about that too. Given the above, and the fact that nearly all mortgages are computed for a 25 year term (but seldom run longer than 5 years, these days), the monthly payment at a 10\% yearly interest rate for an additional $1000 on the mortgage is $8.95. Before you engage in mortgages it would be a good idea to program your spreadsheet with these formulas and convince yourself how bi-weekly payments reduce the total interest you pay, how important a couple of percentage points off the interest rate are to your monthly budget, etc.

### 3.4 Review Problems

**Question 1:** There are three time periods and one consumption good. The consumer’s endowments are 4 units in the first period, 20 units in the second, and 1 unit in the third. The money price for the consumption good is known to be \( p = 1 \) in all periods (no inflation.) Let \( r_{ij} \) denote the (simple, nominal) interest rate from period \( i \) to \( j \).

- **a**) State the restrictions on \( r_{12} \), \( r_{23} \) and \( r_{13} \) implied by zero arbitrage.
- **b**) Write down the consumer’s budget constraint assuming the restriction in (a) holds. Explain why it is useful to have this condition hold (i.e., point out what would cause a potential problem in how you’ve written the budget if the condition in (a) fails.
- **c**) Draw a diagrammatic representation of the budget constraint in periods 2 and 3, being careful to note how period 1 consumption influences this diagram.

**Question 2:** There are two goods, consumption today and tomorrow. Joe
has an initial endowment of \((100, 100)\). There exists a credit market which allows him to borrow or lend against his initial endowment at market interest rates of 0%. A borrowing constraint exists which prevents him from borrowing against more than 60% of his period 2 endowment. Joe also possesses an investment technology which is characterized by a production function \(x_2 = 10 \sqrt{x_1}\). That is, an investment of \(x_1\) units in period 1 will lead to \(x_2\) units in period 2.

a) What is Joe’s budget constraint? A very clearly drawn and well labelled diagram suffices, or you can give it mathematically. Also give a short explanatory paragraph how the set is derived.

b) Suppose that Joe’s preferences can be represented by the function \(U(c_1, c_2) = \exp(c_1^4 c_2^6)\). (Here \(\exp()\) denotes the exponential function.) What is Joe’s final consumption bundle, how much does he invest, and what are his transactions in the credit market.

Question 3: Anna has preferences over her consumption levels in two periods which can be represented by the utility function

\[ u(c_1, c_2) = \min \left\{ \frac{23}{22} \left( \frac{12}{10} c_1 + c_2 \right), \frac{13}{10} c_1 + c_2 \right\}. \]

a) Draw a carefully labelled representation of her indifference curve map.

b) What is her utility maximizing consumption bundle if her initial endowment is \((9, 8)\) and the interest rate is 25%.

c) What is her utility maximizing consumption bundle if her initial endowment is \((5, 12)\) and the interest rate is 25%.

d) Assume she can lend money at 22% and borrow at 28%. What would her endowment have to be for her to be a lender, a borrower?

e) Assume she can lend money at 18% and borrow at 32%. Would Anna ever trade at all? (Explain.)

Question 4: Alice has preferences over consumption in two periods represented by the utility function \(u_A(c_1, c_2) = \ln c_1 + \alpha \ln c_2\), and an endowment of \((12, 6)\). Bob has preferences over consumption in two periods represented by the utility function \(u_B(c_1, c_2) = c_1 + \beta c_2\), and an endowment of \((8, 4)\).

a) Draw an appropriately labelled representation of this exchange economy in order to “prime” your intuition. (Indicate the indifference maps and the Contract Curve.)

b) Assuming, of course, that both \(\alpha\) and \(\beta\) lie strictly between zero and one, what is the equilibrium interest rate and allocation?